Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Behaviour of microscopic black holes

Bachelor Thesis

2022 Matej Hrmo

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Study Programme:	Physics
Field of Study:	Physics
Department:	Department of Theoretical Physics
Supervisor:	Mgr. Samuel Kováčik, PhD.

Bratislava, 2022

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THESIS ASSIGNMENT

Name and Surname:	Matej Hrmo
Study programme:	Physics (Single degree study, bachelor I. deg., full time form)
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Title: Behaviour of microscopic black holes

Annotation: Expected, even though still not observed, property of black holes is the thermal (Hawking) radiation. The temperature of this radiation is inversely proportional to the black hole mass and is basically negligible for stellar black holes. However, in the case of microscopic black holes, the radiation can become intensive. According to the classical temperature, it can reach infinite temperatures.

Various new theories claim, that within the black hole does not reside a perfect singularity, but a blurred one instead (for example Gaussian-blurred). It seems that this modifies the black-hole behaviour greatly.

It has been shown recently that the behaviour is not sensitive to details of the description of the (blurred) singularity inside black holes.

Aim: Goals are:
1. Understand the theoretical background of the work.
2. Propose new mass distribution and analyse the solution.
3. Investigate the generality of behaviour of miscroscopic black holes.

Literature: S. Kováčik, R^3_\Lambda$-inspired black holes, Mod. Phys. Lett. A \textbf{32}, no.25, 1750130 (2017) doi:10.1142/S0217732317501309

S. Kováčik, Hawking-Radiation Recoil of Microscopic Black Holes, (2021), arXiv:2102.06517 [gr-qc]

Comment: This work requires certain amount of numerical computations, learning the basics of black hole theory and analysing underlying equations.

Reading in English and basic programming skills are assumed.

Keywords: Black holes, gravity, singularity

Supervisor:	Mgr. Samuel Kováčik, PhD. (from 2021-09-27)
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Approved: 28.09.2021 prof. RNDr. Jozef Masarik, DrSc.
Guarantor of Study Programme

Student

Supervisor

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Abstrakt

Cieľom tejto práce je skúmať správanie mikroskopickej čiernej diery obahujúcej singularitu, ktorá je rozmazaná ako výsledok možnej kvantovej štruktúry časopriestoru. Práca navrhuje niekoľko takých možných rozmazaných rozložení hmotnosti a následne sa zameriava na rozbor tepelného (Hawkingovho) žiarenia mikroskopickej čiernej diery. Poukazuje na kvalitatívne podobnosti Hawkingovho žiarenia bez ohľadu na detaily rozmazanej singularity vnútri mikroskopickej čiernej diery.

Kľúčové slová: Čierne diery, gravitácia, singularita

Abstract

The aim of this thesis is to investigate the behaviour of a microscopic black hole containing a singularity that is blurred as a result of a possible quantum structure of spacetime. The thesis suggests several such blurred distributions of mass and then focuses on analysing the thermal (Hawking) radiation of a microscopic black hole. It shows a qualitative similarity of the Hawking radiation regardless of the details of the blurred singularity within a microscopic black hole.

Keywords: Black holes, gravity, singularity

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Foreword

Put very briefly, black holes are objects with a mass so high contained in a space so small that not even the light (or anything moving at the speed of light) can escape their gravitational pull.

Black holes are among the more active areas of research in theoretical physics. There remains a lot of mystery surrounding these objects, as observing a black hole is very complicated. A wide range of black holes are currently known to physics, categorized by their mass. It seems, that even microscopic black holes could exist, ones that have a mass so small that their radius would be approximately equal to the Planck length. However, this could have serious consequences for the black hole if there exists a quantum structure of the spacetime (a structure that becomes obvious only at Planck lengths).

The assignment of this thesis prompted the author to attempt to better understand black-hole physics in the context of the general theory of relativity and to explore the behaviour of these black holes. It was predicted by Hawking that a black hole would emit a thermal (black-body-like) radiation, and this radiation — the Hawking radiation — could, in theory, be observed. In the thesis, we aim to explore the universality of this radiation and the microscopic black holes in general. Numerical methods were used in our work to provide a better idea of how the microscopic black hole radiates and to explore its behaviour.

This thesis was chosen by the author after being inspired by this thesis's supervisor's work in the area of microscopic black holes.

Chapter 1

Introduction to black holes

Black holes are a topic enshrouded in mystery and widely popular with general public. However, they are also a target of active research by physicists. With this thesis we shall humbly try to follow in the steps of many, whom explore the uncharted territory of black-hole physics.

This thesis is divided into three chapters — Introduction to black holes, Theoretical motivation, and Results of the work. The first chapter will cover the basic knowledge required for understanding the work. It will also allow the reader to get more acquainted with the topic covered by this thesis — black holes. If the reader feels they have a good understanding of the basics of black-hole physics, it is possible to skip this chapter and continue with Chapter 2.

1.1 Gravitating objects

Let us start from the basics. In the 17th century, Newton formulated his laws of motion as well as the law of universal gravitation. According to this law, every two objects with masses (point masses) M_1 and M_2 respectively are drawn to each other by a certain force with a magnitude given by

$$F_g = G\frac{M_1 M_2}{r^2}, \tag{1.1}$$

where $G \doteq 6.67 \cdot 10^{-11}$ m^3 kg^{-1} s^{-2} is the gravitational constant and r is the mutual distance of those point masses. This force was named the force of gravity.

Let us now imagine two such massive objects at a certain distance, one trying to get away from the other. To make things easier, we can imagine the planet Earth and a spaceship that wishes to leave the planet's surface and escape towards the infinity. There exists a minimal velocity required to achieve such a feat — this velocity is called the escape velocity. Its magnitude is given by various parameters of the situation. If the planet has mass M and the initial distance of the spaceship from the center of the planet (therefore the planet's radius) is R, the escape velocity is given by

$$v_{esc} = \sqrt{\frac{2GM}{R}}, \tag{1.2}$$

which can easily be derived from the conservation of energy. Note that the escape velocity does not depend on the mass of the escaping object.

1.2 Speed of light

In the 17th century, Rømer observed Jupiter's moon Io and observed that the speed of light could not be infinite. A rough finite value was determined, and various measurements were conducted over time to determine this value more accurately.

In the 19th century, Maxwell proposed his equations for the electromagnetic field. Solving these equations leads to discovering, that any electromagnetic wave travels through vacuum at a certain constant speed $c \doteq 2,99 \cdot 10^8$ m s^{-1}. Surprisingly, this speed was equal to the (by then far more accurately) measured speed of light, supporting the theory that visible light is an electromagnetic wave. The speed of light is currently defined as $c = 299\ 792\ 458$ m s^{-1} exactly as to define the unit metre for the SI system.

Einstein's special theory of relativity took this a step further, postulating that the speed of light (in vacuum) is constant with respect to every inertial reference frame and that it is the maximal velocity at which anything can travel through space.

1.3 What is a black hole?

We will now take a semi-classical approach to obtain some intuition on what a black hole is. Let us note that the current view of black holes includes the general theory of relativity.

Let us now imagine a sphere of radius R and mass M. Looking at (1.2), we can see that the higher the mass and the smaller the radius, the greater the escape velocity of such a sphere would be. Let us therefore imagine that we have such a sphere and we will compress it to have a smaller radius, keeping the mass constant. As we keep compressing, there must exist a certain limit for the radius when the escape velocity is equal to the speed of light c. This radius is known as the Schwarzschild radius, denoted R_S, and can easily be derived from (1.2) as

$$R_S = \frac{2GM}{c^2}. \tag{1.3}$$

It is a radius that a sphere containing a mass M would have to have in order to have the escape velocity (from its surface) equal to the speed of light. Let us note that the mass can, in fact, be easily compressed into a smaller sphere. We then do not speak of a true surface, but instead, it is the limit distance from which light can escape. The points of space with such distance from the centre of the space are called an event horizon. Points of space inside the event horizon are called a black hole. In the classical (Newtonian) sense, it is a place that has greater escape velocity than the speed of light, making it impossible for anything to escape from the black hole. As not even light can escape from it, it was given the name "black hole" (in the the late 60's by J. A. Wheeler).

These objects need to have a very, very high density. To give the reader a better idea of this density, let us provide an example. The planet Earth approximately has a mass $M_{\oplus} \doteq 5,97 \cdot 10^{24}$ kg and a radius $R_{\oplus} \doteq 6,38 \cdot 10^{6}$ m. Using (1.3), we obtain that should we wish to create a black hole out of the whole planet Earth, we would need to compress it to a radius of approximately 9 millimetres. For our Sun with the mass $M_{\odot} \doteq 1,99 \cdot 10^{30}$ kg (the so-called solar mass) and radius $R_{\odot} \doteq 6,96 \cdot 10^{8}$ m, the Schwarzschild radius would be approximately 2953 metres.

We can therefore see that black holes are incredibly dense — either very small for their mass or absurdly massive for their radius.

1.3.1 A singularity inside a black hole

As mentioned above, the view of black holes provided thus far is more classical or semi-classical. Let us, at least very briefly, mention the main difference between the classical view and the modern view.

In 1915, Einstein published the Einstein field equations, which describe the geometry of curved spacetime in the presence of mass. This theory allowed an existence of objects that would create regions from which nothing could escape. A few months later, in 1916, Schwarzschild provided a spherically-symmetric solution (which we will mention in Chapter 2 of this thesis) to these equations in vacuum. This solution, however, had a slight (should we choose to call it that) flaw — it was continuous everywhere but in the origin of spatial coordinates. The topic reopened in 1939 when Oppenheimer and Snyder calculated the collapse of a sphere of pressure-less gas, showing that the matter would compress to a single point, creating infinite density in the centre — a singularity. The work on black holes was then continued by Wheeler in the late 60s.

To enable the reader to understand black holes better, we advise reading for example [7].

Chapter 2

Theoretical motivation

In the second chapter, we will provide a more detailed insight into the work. We will also introduce the proposed matter densities (see below), which will be crucial for our work.

Lecture notes by Sean M. Carroll [3] and the book by Marián Fecko [5] have helped the author to understand the mathematics behind general relativity better — in these sources, more details can be found.

2.1 Black holes and the Schwarzschild solution

The existence of black holes was predicted by the general theory of relativity, where such objects would be a valid solution to the Einstein's equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{2.1}$$

These are 16 equations altogether, each equation is obtained by plugging one (t,r,ϑ,φ) into the each of the indices on both side, yielding 16 combinations. The equations contain two objects. On the left-hand side there is the Einstein tensor G, for which

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{2.2}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar and $g_{\mu\nu}$ is the metric tensor. The left-hand side therefore defines the curvature of the spacetime. On the right-hand side, there is the stress-energy-momentum tensor (its covariant version to be completely exact) $T_{\mu\nu}$, which describes the density of and flux of energy and momentum in the spacetime. Most importantly (in our context), the T_{tt} component is the matter density distribution present in the spacetime.

Note that from equation (2.1) on, unless explicitly stated otherwise, throughout this work we will be adjusting our units so that $G = c = k_B = 1$ to simplify the results.

Arguably the most important such solution is the Schwarzschild solution of Einstein field equations in a vacuum spacetime, which assumes spherical symmetry — this results in a specific ansatz for the metric tensor. Using coordinates (t, r, θ, φ) and choosing signature $(-, +, +, +)$, the metric tensor would take the form of

$$g = \mathrm{diag}(f(r), -1/f(r), r^2, r^2 \sin\theta) \tag{2.3}$$

for some unknown function $f(r)$. We shall use this ansatz for the metric tensor in our work as well. Solving one of the Einstein field equations, for example

$$G_{tt} = 8\pi T_{tt}, \tag{2.4}$$

with such metric tensor would yield the unknown function $f(r)$ if we knew the right hand side. We arrive at the equation

$$\frac{1 + f(r) + rf'(r)}{r^2} = 8\pi\rho(r), \tag{2.5}$$

where $\rho(r)$ is the matter density of the black hole. The equation (2.5) is an ordinary differential equation, that for a given matter density is solved by the integral form

$$f(r) = -1 + \frac{2M}{r} \int_0^r \tilde{\rho}(r) 4\pi r^2 \mathrm{d}r, \tag{2.6}$$

where ρ_n is a normalized matter density and M is the mass of the black hole, so that $\rho(r) = M\tilde{\rho}(r)$ and

$$\int_0^\infty \tilde{\rho}(r) 4\pi r^2 \mathrm{d}r = 1. \tag{2.7}$$

We can see that the unknown function f(r) can easily be obtained by plugging in a chosen matter density $\rho(r)$ and calculating the integral given by (4). Choosing to use the Dirac delta-distribution as our $\rho(r)$ — a singularity in the centre of our black hole — we arrive at the Schwarzschild solution,

$$f(r) = -1 + \frac{2M}{r}. \tag{2.8}$$

We can see from 2.1 that the event horizon is closer to the centre of the black hole for smallest the mass (blue) and further away for the largest mass (violet).

For future needs, let us note that, in our signature, the black hole is a region where the purely temporal component of the metric tensor, g_{tt} (which is our function $f(r)$), is positive. We find an event horizon at those points of spacetime where

$$f(r) = g_{tt}(r) = 0. \tag{2.9}$$

We can easily see from (2.9) that the Schwarzschild solution provides us with a single event horizon at $r = 2M$.

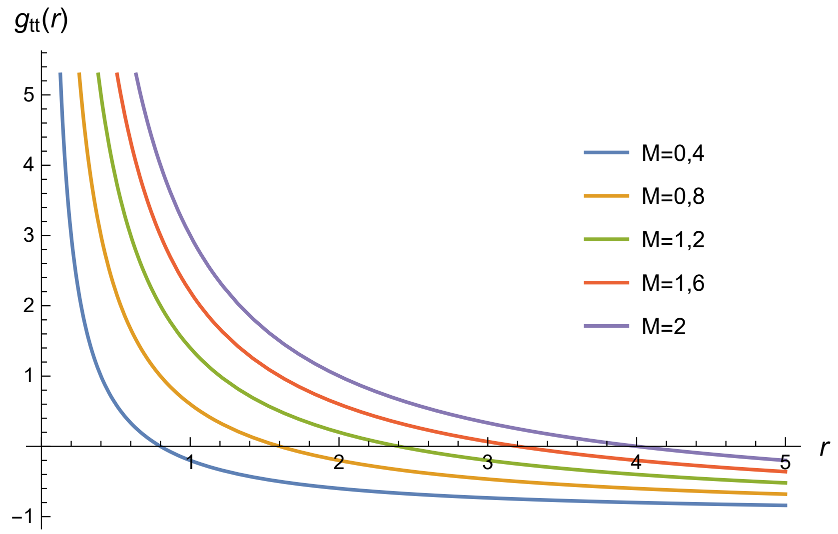


Figure 2.1: Schwarzschild solution for various masses in Planck units.

2.2 Influence of a quantum structure

Theoretical physicists have worked with the concept of a spacetime that would have some sort of a structure on a very small (quantum) scale — possibly the Planck scale — for quite some time now. Details of such structure vary from theory to theory, but what is shared by the theories is the absence of infinitely short distances.

This would, however, mean that the Dirac delta distribution is not a good candidate for the matter density within a black hole — because the Dirac delta-distribution would not be present in a spacetime where we cannot distinguish two points that are a certain length λ apart. We would have to use some other matter density function, possibly only very slightly different from the Dirac delta distribution — a so-called blurred singularity. This concept has been explored by scientists in the past (see [2], [4], [8], [9], [10], [11], [13]). One could argue that this effect is negligible for a black hole that has a radius $r \gg \lambda$. Let us, however, imagine a black hole that has a radius comparable to this length scale, $r \approx \lambda$ — a so-called microscopic black hole. For such an object, the change from a perfect singularity to a blurred one is shown to have various interesting consequences, such as that there could be two event horizons (one within the other) present, and there could even be more than two. As a part of this work, we will investigate the qualitative influence of changing the blurred singularity on the purely temporal component of the metric tensor.

Of course, any quantum structure should not have any significant impact on a black hole with a radius $r \gg \lambda$. Therefore, as the mass of the black hole increases, the difference between a perfect and blurred singularity should become less significant and we should observe the (outermost) event horizon at a position given by the Schwarzschild solution (2.8).

2.3 Hawking radiation

It shows, that the black holes may not be so "black" after all. At first, we thought that nothing could be coming our way from a black hole, but this turned out wrong. Combining quantum field theory and curved spacetime it was shown by Hawking that a black hole should indeed radiate [6].

Let us attempt to provide an intuitive view of how a black hole can radiate. First, let us start with quantum fluctuations of spacetime. It is a phenomenon where a pair particle-antiparticle is spontaneously created for a short time. The particle and antiparticle each travel a short path and then annihilate. However, should such pair emerge in close proximity to an event horizon of a black hole, there is a possibility that half of the pair falls into the black hole (from whence it cannot escape) and the other half escapes towards infinity.

The particle that escapes must carry some (positive) energy, which, due to energy conservation, means that some energy is taken from the black hole — as if something with negative energy would fall into the black hole. If we interpret this as a particle with negative mass, the phenomenon can be viewed as if the black hole radiated a particle with positive energy and mass, losing its own mass in the process. The particles leaving the black hole can be interpreted as thermal radiation, known as Hawking radiation. It is one of the possibly observable behaviours of a black hole.

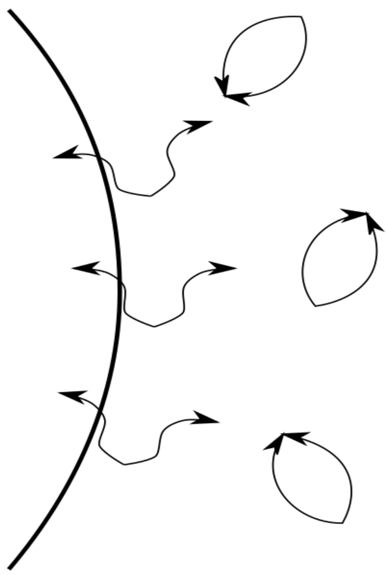


Figure 2.2: Hawking radiation as a result of quantum fluctuations near the event horizon.

Let us note that it has been shown [12], that the particle-antiparticle pair may be created under the event horizon and one half of the pair may escape via quantum tunneling.

It has been shown [6], that the temperature of the Hawking radiation is proportional to the surface gravity of the black hole κ. The surface gravity of the black hole is equal to (see [6])

$$\kappa = \frac{-g'_{tt}(r_0)}{2} = \frac{-f'(r_0)}{2}, \tag{2.10}$$

where r_0 is the position of the event horizon — this would, of course, mean the outermost event horizon, should there be multiple event horizons (which we will show later). Specifically, the Hawking temperature itself can be found as

$$T = \frac{\kappa}{2\pi} = \frac{-f'(r_0)}{4\pi}. \tag{2.11}$$

This shows that, since choosing a specific blurred singularity as our matter density fully defines the metric tensor (see (2.3) and (2.6)), the specifics on how the density is blurred will affect the Hawking temperature, which is (at least theoretically) an observable behaviour of the microscopic black hole.

2.4 Proposed matter densities

Recent research [2], [4], [9], [11] investigates the behaviour of both the purely temporal component of the metric tensor and Hawking radiation of a microscopic black hole for several classes (with a free parameter n) of matter densities given by

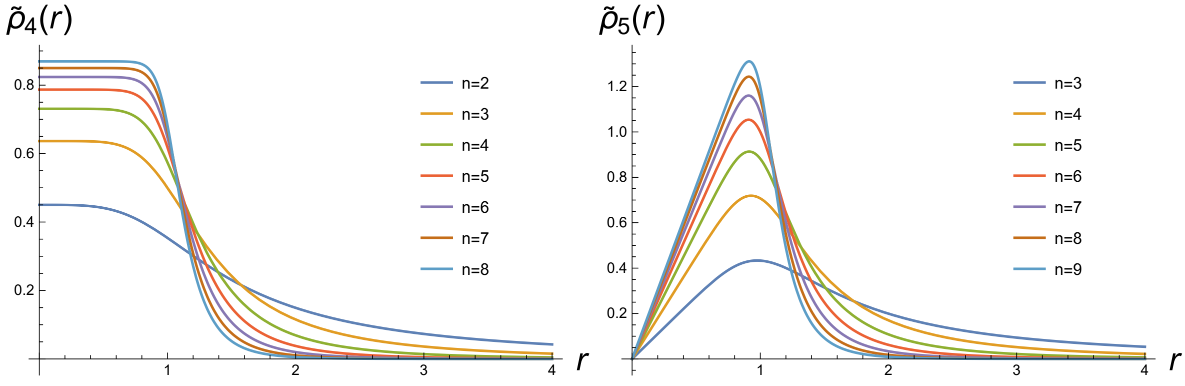
$$\begin{aligned} \tilde{\rho}_1(r;n) &= C_1 \cdot \mathrm{e}^{-(r/\lambda)^n} \\ \tilde{\rho}_2(r;n) &= C_2 \cdot (1 + (r/\lambda))^{-n} \\ \tilde{\rho}_3(r;n) &= C_3 \cdot (1 + (r/\lambda)^n)^{-1} \end{aligned} \tag{2.12}$$

The shared property of these matter densities is that they rather rapidly tend to zero and attain a maximum at zero, well modeling a blurred singularity. In this work we shall propose two other matter densities. The chosen classes of matter densities are given by

$$\tilde{\rho}_4(r;n) = C_4 \cdot \frac{\arctan{(r/\lambda)^n}}{(r/\lambda)^n} \tag{2.13}$$

$$\tilde{\rho}_5(r;n) = C_5 \cdot \frac{\arctan{(r/\lambda)^n}}{(r/\lambda)^{n-1}} \tag{2.14}$$

The density given by (2.14) does not share the maximum property and instead will grow from zero towards a maximum and then rapidly tend to zero as well. We will analyze the metric tensor and Hawking temperature for the proposed densities.



(a) $\tilde{\rho}_4(r;n)$ (b) $\tilde{\rho}_5(r;n)$

Figure 2.3: Proposed classes of matter density for several values of the parameter n each.

We observe that as the parameter n increases, the width of the "peak" in both matter density classes decreases. Henceforth unless said otherwise, we shall set $\lambda = 1$, as if making our r dimensionless, to simplify the results.

We can see that the proposed classes of matter density are quite different. However, the Hawking temperature profile, as we shall show in the following chapter, proves to be very similar despite the differences in matter density distributions.

Chapter 3

Results of the work

In the third chapter of this thesis, we shall present the analysis of results for the proposed densities (2.13) and (2.14). We will obtain the solution $f(r)$ for every matter density and then observe the Hawking temperature profile. To aid us in those parts of our work that need to be done numerically, we use `Wolfram Mathematica 11.3` computational software.

3.1 Analyzing solutions

The very first step we need to take is calculating $f(r)$ (as given by (2.6)) for our proposed densities, which at first may simply seem to be an integral of a relatively simple function. However, the results show to be a little more complicated. If we denote $f_i = -1 + \frac{2M}{r} C_i \vartheta_i(r)$, we will find the integral parts of $f_i(r)$ as

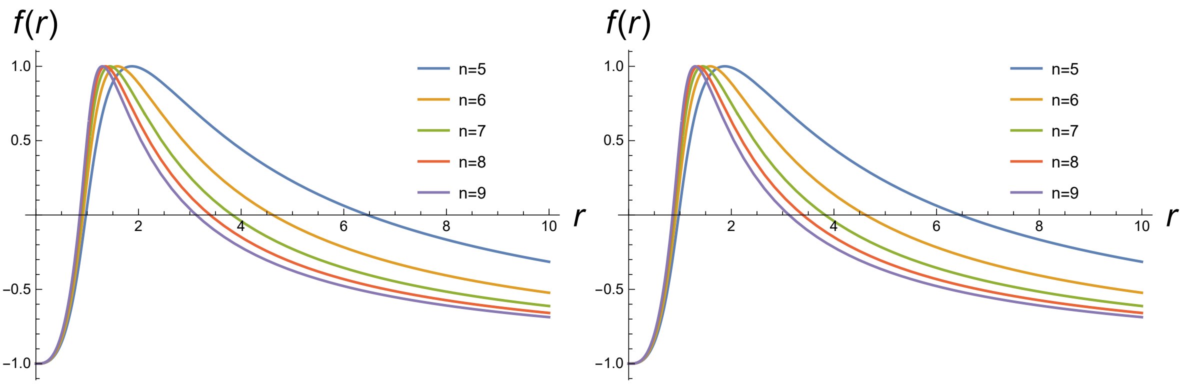
$$\vartheta_4(r) = \frac{4\pi r^3 \left(n - 3r^{-n} \arctan\left(r^n\right) - n \;{}_s2F_1\left(1, -\frac{3}{2n}; 1 - \frac{3}{2n}; -r^{2n}\right)\right)}{3\left(n-3\right)}, \tag{3.1}$$

$$\vartheta_5(r) = \frac{\pi r^4 \left(n - 4r^{-n} \arctan\left(r^n\right) - n \;{}_2F_1\left(1, -\frac{2}{n}; \frac{n-2}{n}; -r^{2n}\right)\right)}{n-4} \tag{3.2}$$

where ${}_2F_1$ is the Gaussian hypergeometric function (see [1]).

The integral parts of the solution do not converge for every $n \in \mathbb{N}$. This is because of that we require the existence of $\int_0^\infty \tilde{\rho}_i(r;n) 4\pi r^2 \mathrm{d}r$. This integral does only converge for $n > 3$ or $n > 4$ if we use $\tilde{\rho}_4(r;n)$ or $\tilde{\rho}_5(r;n)$ respectively.

Figures 3.1a and 3.1b each show the $f_i(r)$ for the first five parameters n for which the solution exists respectively. They show the qualitative similarities of the solutions, both within one class and among each other. We can see that in every of these cases, $f(r)$ is a function complying to $f(0) = -1$ and we can see that for $r \to \infty$, the solution also tends to -1 — this is also obvious from (2.6, as our solution should in this limit match Schwarzschild's. Between $r = 0$ and $r \to \infty$ the function always reaches a single maximum, henceforth denoted r_0. The position of this maximum is defined by the θ_i part of the solution and is completely independent of the total mass M. Whether this maximum is positive, zero or negative depends on the mass M that the microscopic black hole possesses. There exists a certain critical mass (different for every solution), henceforth denoted M_0, for which the $f(r_0) = 0$, making r_0 an event horizon. If $M < M_0$, then there is no event horizon (and therefore no black hole), if $M > M_0$, we can see that there will always be two event horizons, one within the other, at positions henceforth denoted r_{in} and r_{out}. Figures 3.1a and 3.1b were plotted with mass of $2M_0$ for every separate solution $f(r)$.



(a) $f_4(r)$ for $n = 4, 5, 6, 7, 8$

(b) $f_5(r)$ for $n = 5, 6, 7, 8, 9$

Figure 3.1: Solutions $f(r)$ for the proposed densities.

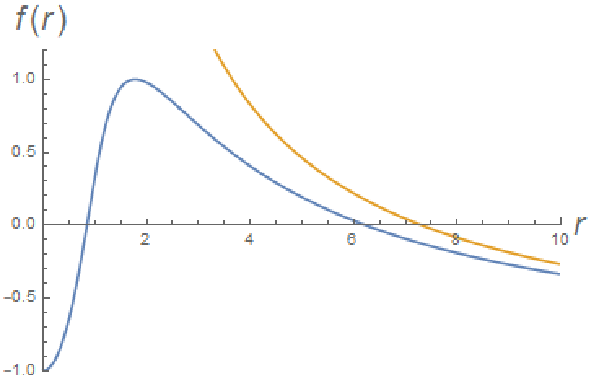
We observe that the peak of $f(r)$ narrows as the parameter n increases for these functions.

At this point, we would like to note that these functions, $f_4(r; n)$ and $f_5(r; n)$, were generated by matter densities that share some common properties — they reach a single maximum and rapidly tend to zero. Therefore, the aforementioned properties of the solutions may be (and, as we later show, indeed are) a result of the shared properties of the matter density classes.

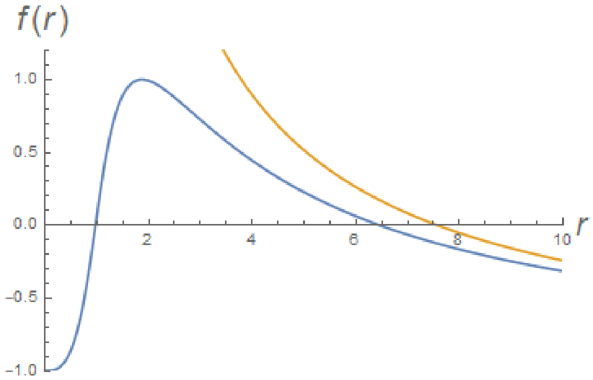
3.2 Comparison to Schwarzschild solution

Before moving to explore the Hawking radiation of a microscopic black hole with the proposed metric tensor, we will take a while to discuss whether our solutions did not stray too far from the Schwarzschild solution. We would wish for our solutions to provide an event horizon (the outer one) at the position given by the Schwarzschild radius, or at least nearby.

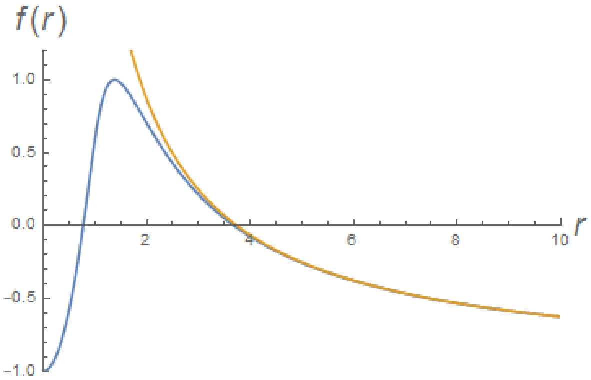
To see how our solutions fare with respect to the Schwarzschild solution, we will plot selected solutions for both classes of matter density.



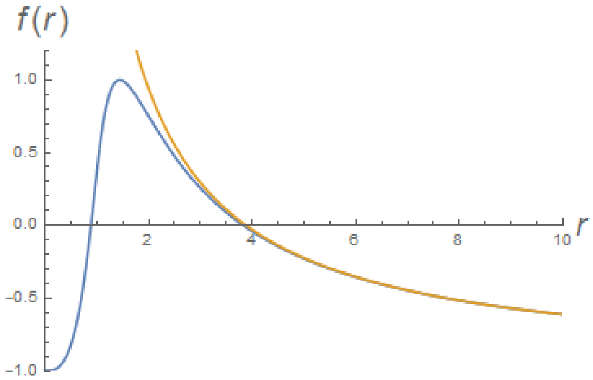
(a) $f_4(r;4)$



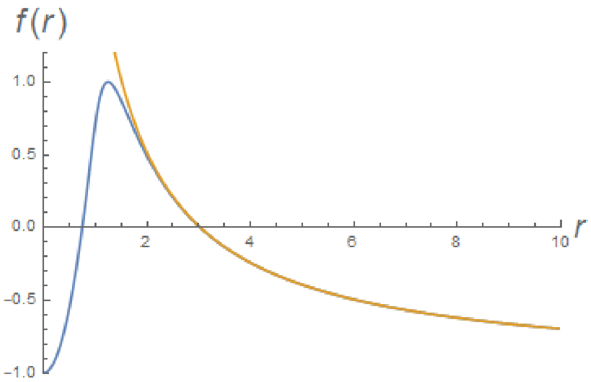
(b) $f_5(r;5)$



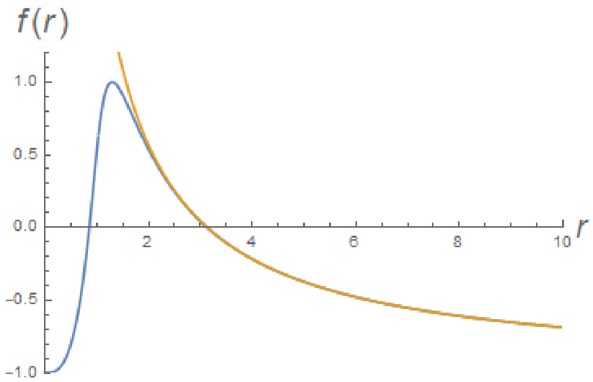
(c) $f_4(r;6)$



(d) $f_5(r;7)$



(e) $f_4(r;8)$



(f) $f_5(r;9)$

Figure 3.2: Comparison to Schwarzschild solution.

In figures 3.2a-f we can see that as the parameter n increases, the solutions for both classes of matter density start to match the Schwarzschild solution quite well. We can, however, also see that for the lowest value of the parameter possible in each class, our solutions do not match the Schwarzschild solution near the event horizon. Luckily, this difference becomes negligible if we increase the mass of the black hole and therefore our solutions will still resemble the Schwarzschild solution very well for a non-microscopic black hole.

3.3 Hawking temperature profiles for our matter densities

The next step in our work is to analyse the Hawking radiation of a microscopic black hole – meaning that we will analyse the Hawking temperature (given by (2.11)) of the microscopic black hole with a proposed matter density with respect to its total mass M.

Were we working with the possibility of a perfect singularity (and therefore the Schwarzschild solution), the black hole could radiate particles and theoretically reach an unlimitedly small radius (and mass). The Schwarzschild solution provides us with a single event horizon given by the Schwarzschild radius $r_s = 2M$. Substituting the Schwarzschild solution (2.8) into (2.11) leads to

$$T(M) = \frac{1}{8\pi M}, \tag{3.3}$$

we can therefore clearly see that as the black hole evaporates, its temperature grows infinitely.

It has been shown (see [9]) that for a matter density given by (2.12), the Hawking temperature does not grow infinitely as the mass decreases. Instead, it vanishes for a certain mass. This property is shared by the solutions given by our proposed matter densities. It is caused by the fact that as the microscopic black hole evaporates particles, it reaches the critical mass M_0 with a single event horizon situated at r_0 and as $f'(r_0)$ vanishes, the Hawking temperature of a black hole with this critical mass vanishes as well. We are left with what is called a microscopic black hole remnant.

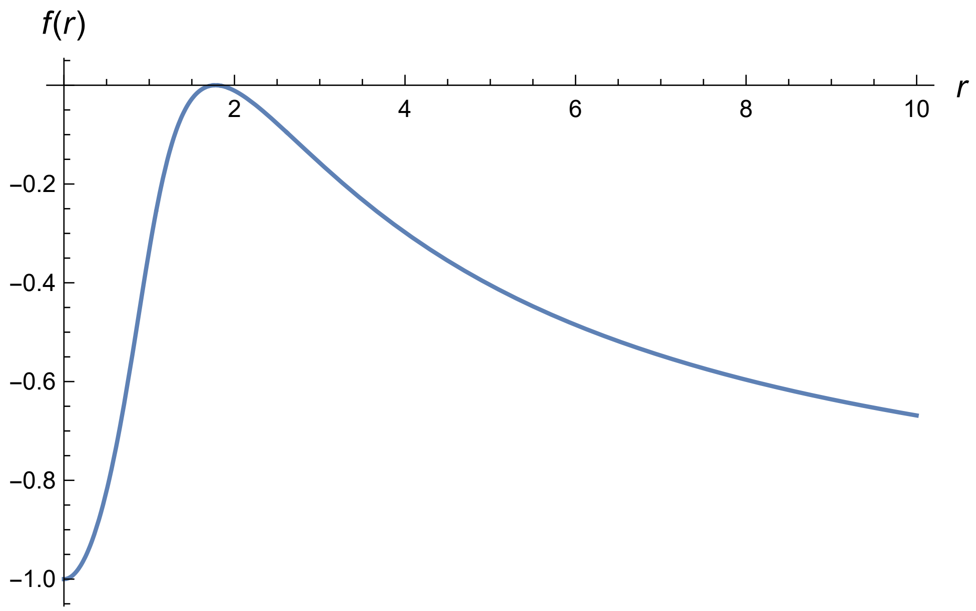
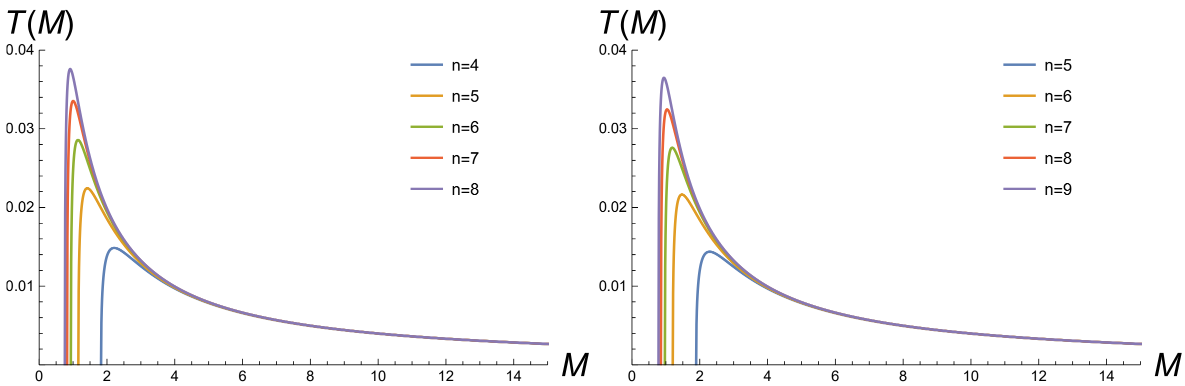


Figure 3.3: Microscopic black hole remnant for $f_4(r;4)$.

Knowing this, we can easily obtain the temperature profile $T(M)$ of a microscopic black hole numerically. We know that $T(M_0) = 0$ and we can keep adding small bits of mass dM and see what the Hawking temperature given by (2.11). For that, of course, we need to find the outer event horizon r_{out}, which will be the greater solution to the equation $f(r) = 0$. As we have hinted above, its position will depend on the mass M. Then we calculate the derivative $f'(r_{out})$ in the event horizon to find the Hawking temperature — this derivative does also depend on the mass M, as the mass dictates the steepness of $f(r)$. This way, we numerically obtain a set of points $(M, T(M))$ that we can plot.



(a) Temperature profiles given by $f_4(r;n)$ for $n = 4, 5, 6, 7, 8$

(b) Temperature profiles given by $f_4(r;n)$ for $n = 5, 6, 7, 8, 9$

Figure 3.4: Temperature profiles given our solutions in Planck units.

Should we do so, we will notice that $T(M)$ also reaches a single maximum. The value of this maximum increases from the lowest value of parameter n (blue) to the highest (violet) for both solution classes. We shall denote this maximum T_{max} and the mass when the microscopic black hole reaches this temperature we shall denote M_1. We can see that the qualitative description of the temperature profile is always the same. The temperature steeply rises from zero (at M_0) to a maximum (at M_1) and then tends slowly to zero again. For this reason, we may obtain the idea to re-scale our plots so that the point (M_1, T_{max}) is the same point for every profile. We obtain the following figures.

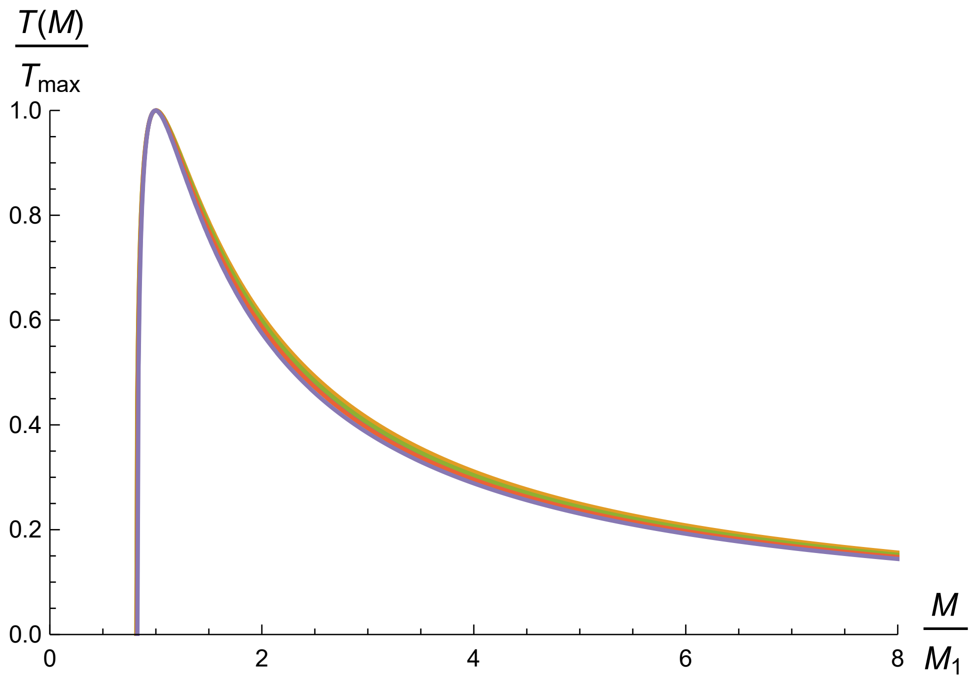


Figure 3.5: Re-scaled temperature profiles given by $f_4(r;n)$ for $n = 4, 5, 6, 7, 8$.

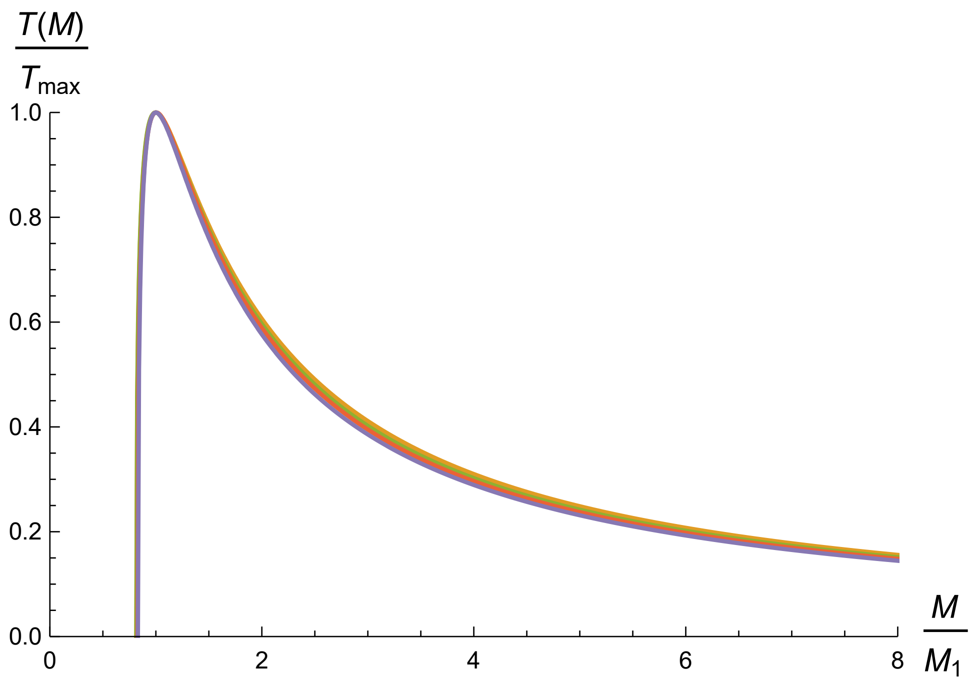


Figure 3.6: Re-scaled temperature profiles given by $f_5(r;n)$ for $n = 5, 6, 7, 8, 9$.

We can see that these figures are very, very similar and that the temperature profiles for our solutions are nearly identical, despite the fact that the original matter densities were quite different (see Figure 2.3). This may lead us to thinking that there is some universality to this result. To further support this claim, we provide a comparison between our proposed densities and the matter density class $\tilde{\rho}_1(r;n)$ proposed by [9]. We can see that they all have the same qualitative description mentioned above, differing only very slightly.

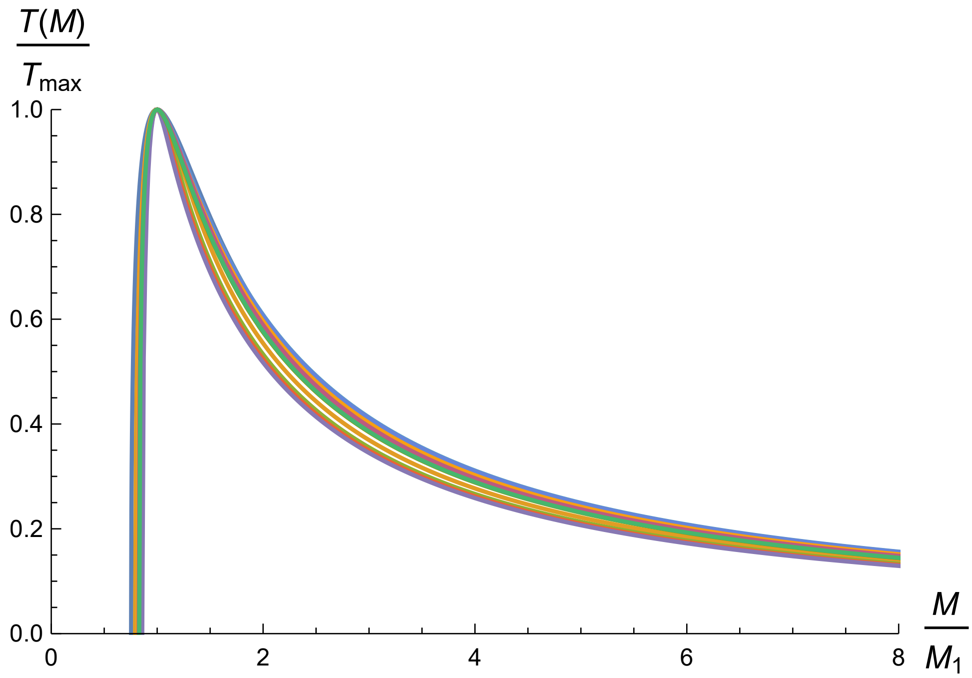


Figure 3.7: Comparison of our temperature profiles with those given by $\tilde{\rho}_1(r;n)$.

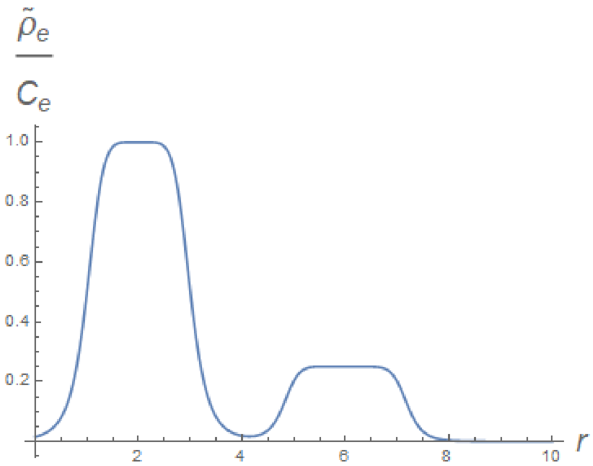
However, all these temperature profiles were obtained from matter densities that provide us with maximally two event horizons.

3.4 Case of more event horizons

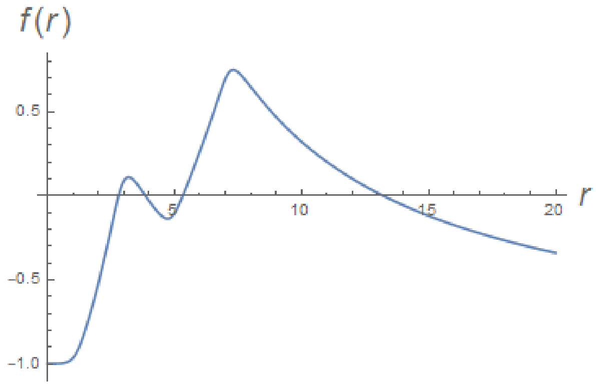
In this section, we will analyze how the results of our work — especially the Hawking radiation — change for a matter density that would allow the microscopic black hole to have more than two event horizons. The goal is to investigate how the chosen matter density $\tilde{\rho}(r)$ impacts the solution $f(r)$, which then defines the behaviour of the microscopic black hole. As an example of such matter density, we can choose

$$\tilde{\rho}_e(r) = C_e \left(\left(1 + (r-2)^6\right)^{-1} + \left(4 + (r-6)^8\right)^{-1} \right) \tag{3.4}$$

The constant C_e is relatively small and fixed by normalisation. Therefore, to give the reader a clearer idea of how this matter density looks, in the following figure we will show it without the normalisation constant. We will as well show the solution $f(r)$ for this matter density. We will conveniently choose such total mass that the number of event horizons is greater than two.



(a) $\tilde{\rho}_e(r)$ without normalisation constant.



(b) Solution $f(r)$ given by $\tilde{\rho}_e(r)$.

Figure 3.8: A matter density that yields four event horizons for a certain mass.

We can see that if we choose a normalised matter density that only shares the tend-to-zero property, we may obtain a solution that has more than one local maximum and therefore for a certain mass will have more than two event horizons — one outer and several inner horizons. The surface gravity and the Hawking temperature, of course, are calculated at the outer horizon.

The question we are left with is — what would the temperature profile look like for such an odd solution that allows more than two event horizons? Locating the outermost event horizon (also denoted r_{out}) and finding the mass M_0 such that $f(r_{out}) = 0$, repeating the whole procedure, we obtain the Hawking temperature profile for this solution as well.

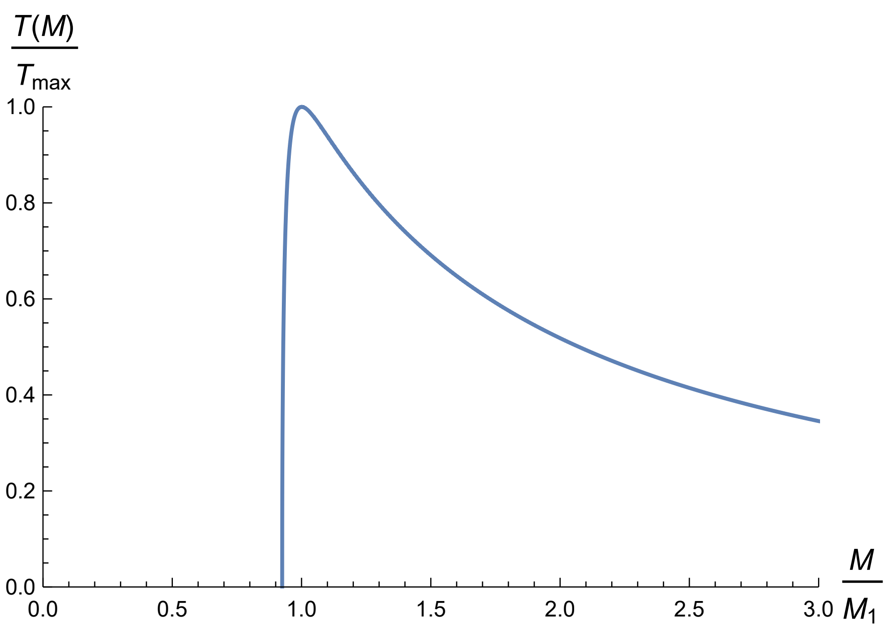


Figure 3.9: Hawking temperature profile given by $\tilde{\rho}_e(r)$.

To our surprise, we can see that this temperature profile is similar to the ones we obtained before. This further supports the conjecture that the qualitative description of the Hawking temperature profile is something more fundamental, something that does not depend on the details of the matter density within the microscopic black hole.

3.5 Further steps

There is still a lot of unknown concerning the topic of microscopic black holes and their behaviour that would require further research. In this section, we aim to provide a reader with a few ideas of what could the next steps be and some questions that the author was not able to answer yet.

3.5.1 Quantitative results

The work that we have done on the Hawking temperature profiles is done numerically (see Appendix A for our algorithm for finding these profiles). Let us therefore provide a reader with a table of numerical values. For each shown matter density, we will provide the masses M_0 and M_1, the event horizon distance r_0 for the critical mass, as well as the maximal Hawking temperature T_{max}. Let us remind the reader that we use the Planck units defined in section 2.1 and that we still work with $\lambda = 1$ here.

Numerical values				
	r_0	M_0	M_1	T_{max}
$\tilde{\rho}_4(r;4)$	1.777	1.825	2.216(1)	0.015(1)
$\tilde{\rho}_4(r;5)$	1.509	1.156	1.416(1)	0.022(1)
$\tilde{\rho}_4(r;6)$	1.377	0.933	1.144(1)	0.029(1)
$\tilde{\rho}_4(r;6)$	1.298	0.823	1.003(1)	0.034(1)
$\tilde{\rho}_4(r;8)$	1.247	0.757	0.917(1)	0.038(1)
$\tilde{\rho}_5(r;5)$	1.871	1.894	2.294(1)	0.014(1)
$\tilde{\rho}_5(r;6)$	1.159	1.203	1.473(1)	0.022(1)
$\tilde{\rho}_5(r;7)$	1.442	0.970	1.190(1)	0.028(1)
$\tilde{\rho}_5(r;8)$	1.354	0.853	1.033(1)	0.033(1)
$\tilde{\rho}_5(r;9)$	1.295	0.783	0.943(1)	0.037(1)

Table 3.1: Numerical values for our proposed matter densities

Note that the uncertainties given in the table are caused by numerical searching for the event horizon for the given mass (see Appendix A) using the square of left-hand side method. The rest of the values are obtained with `Wolfram Mathematica 11.3` default `AccuracyGoal` which would allow precision up to 8 digits. These results are, of course, rounded to the same precision as the other results.

We can see clearly from Table 3.1 that as the parameter n increases, the critical mass M_0 decreases, but the Hawking temperature increases.

The next step in this area would be to calculate the exact function defining the temperature profile for any of the proposed matter densities. However, this task proves to be quite difficult as with the increasing mass of the black hole, the position of the outer event horizon increases, and the steepness of $f(r)$ at the outer horizon also increases. Combining these two factors proves complicated and therefore we have chosen a numerical approach instead.

3.5.2 Observation of microscopic black holes

Our proposed matter densities lead to microscopic black holes that radiate (in peak) at nearly Planck temperatures ($10^{-2}T_{\mathrm{P}}$) and it is a radiation with very high energy ($10^{-2}E_{\mathrm{P}}$) As mentioned by [10], the radiation with the highest observed energy are gamma-ray bursts, with energy approximately $10^{-9}E_{\mathrm{P}}$. Not even increasing the parameter n will aid in our case, because then the maximal temperature would only increase! A possibility lays in increasing the length scale λ. According to [8], the maximal temperature scales with λ^{-1}, therefore we would have to increase the length scale by 7 orders to not go above the currently maximal energy of radiation measured.

Another option is to search for more matter densities, selecting one that would naturally fit the observed gamma-ray bursts even at a Planck-length scale.

3.5.3 More than two event horizons

As we have suggested in section 3.4, there exist matter densities $\tilde{\rho}(r)$ that will result in a solution $f(r)$ with more than two event horizons. This bids two interesting questions.

The first question is, how many event horizons can a black hole have in total? If we use the Schwarzschild-like metric tensor (2.3), it is apparent that the maximal possible number of event horizons (for any total mass of the black hole), in this subsection denoted h must be defined by the matter density $\tilde{\rho}(r)$ that resides within the black hole. Our suggestion is that the value of h depends on the number of maxima that $\tilde{\rho}(r)$ has.

The assumption is that the necessary condition for a black hole to have more than two event horizons is that the defining matter density itself has more than one maximum — with this idea we created the matter density $\tilde{\rho}_e(r)$ mentioned in previous section. It is however easy to show that this is not a sufficient condition — let us define another matter density

$$\tilde{\rho}'_e = C'_e \cdot \left(1 + \left((r-3.4)^6\right)^{-1} + \left(4 + (r-6)^8\right)^{-1}\right) \tag{3.5}$$

We can see that (3.4) and (3.5) differ only very slightly — in the position of the first maximum. They both have two local maxima. The difference is however enough so that solution for (3.5) has $h = 2$. It is not easily seen if we plot the solution itself, however if we compute the r-derivative of the solution, we can more easily see there is only one maximum for $f(r)$. This implies there may only be two event horizons (as the solution tends to -1 in both zero and infinity) for any mass. This, however, means that two maxima in the matter density distribution do not imply the possibility of more than two event horizons.

The second question is, how will the number of event horizons affect the Hawking temperature profile? This is a crucial question, however a hard one to answer. As we have mentioned above, we currently have no good quantitative way to describe the Hawking temperature profile. We assume, however, that the Hawking temperature profile will not depend on how many inner event horizons there are inside the outer one. It is a bold assumption, but the Hawking temperature is defined only by the steepness of the solution in the outer event horizon and this steepness is defined at most by the last peek of the solution. If such assumption is true, then (speaking purely theoretically) were we to detect radiation matching this temperature profile, we could maybe speak of observing a microscopic black hole.

Conclusion

It remains to conclude the results of our work. In this thesis, we have continued the exploration of the behaviour of microscopic black holes. We have proposed two new matter densities. The proposed matter densities differed in a single aspect (number of minima), allowing us to explore the impact of this aspect on the behaviour of a microscopic black hole.

We have analysed the solutions given by the Einstein field equations (using a Schwarzschild-like metric) for our proposed matter densities, discovering a certain qualitative similarity — the solutions were functions with one maximum, allowing at most two event horizons for the microscopic black hole. Assuming that this qualitative similarity is caused by the number of maxima in the proposed matter density, we have explored the impact of having more maxima present in the proposed matter density, showing that this may but also may not lead (depending on how close these maxima are) to the microscopic black hole being able to have more than two event horizons for a certain total mass.

We have also compared the originally proposed matter densities to the widely known Schwarzschild solution, showing that even for small masses, the results (position of the outer/only event horizon, behaviour of the solution past the event horizon towards infinity) match quite well. This means that the solutions for our proposed matter densities do not significantly contradict the Schwarzschild view of a black hole.

Next, we have analysed the Hawking temperature profiles for the solutions obtained from the proposed matter densities. Employing numerical methods, we have shown that all the Hawking temperature profiles share a qualitative universality assumed by the previous research in this area — the temperature steeply rises from zero for a certain critical mass towards a single maximum and then more slowly drops again towards zero. We have also explored whether this temperature profile's qualitative description does depend on how many inner event horizons there are within the outer one. For the chosen solution with four event horizons, we have shown that the Hawking temperature profile remains qualitatively similar.

Lastly, we have proposed more unanswered questions in this area. This may allow any willing physicist to continue this work. The proposed questions focus on the areas that we have not been able to explore satisfyingly due to, mostly, mathematical complexity. The questions are: how to obtain a quantitative description of the Hawking temperature profiles and how many event horizons there may be present in total for a microscopic black hole and their effect on its behaviour? We view it to be meaningful to search for answers to these questions, as we believe that they are the key to better understanding the nature, behaviour and significance of microscopic black holes.

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Appendix A: Numerical algorithm used to plot Hawking temperature profiles

In Appendix A, we will provide some more details on how to obtain the Hawking temperature profile numerically. It will use the same notation as the thesis does until here, with an index A to note it is just an example for this Appendix.

We start with a non-normalised matter density $\tilde{\rho}(r) = \frac{\tilde{\rho}_A(r)}{C_A}$. Proposing the matter density is usually done without normalisation, therefore we calculate the normalisation constant C_A using

$$C_A = \frac{1}{\int_0^\infty 4\pi r^2 \tilde{\rho}(r)}. \tag{A.1}$$

We now hold a normalised matter density $\tilde{\rho}_A(r)$. We can calculate the solution $f_A(r)$ using (2.6), choosing $M = 1$.

Having the solution with mass $M = 1$ ready, we numerically find the maxima of this function. If we find multiple (which is not the case for our proposed densities), we are concerned with the last one. This last maximum is the position r_0 of the outer horizon would the black hole have a critical mass M_0.

The next step is calculating the critical mass M_0, which can simply be done by solving the equation

$$(f_A(r) + 1) \cdot M_0 - 1 = 0, \tag{A.2}$$

for M_0.

Now we are prepared to start obtaining the Hawking temperature profile. This will be done by plotting a set of points $(M, T(M))$. We start from point $(M_0, 0)$. Choosing a step $\mathrm{d}M$ we will add a small bit of mass to the black hole and calculate the temperature step by step in a cycle. In the body of the cycle we execute the following:

- find the position of the event horizon r_1,
  - this is done by solving the equation (A.2) with the current mass instead of M_0, solving however for r instead. Should we encounter problems, we can

instead look for a minimum of the function that is defined by the square of the left hand side of (A.2) in a small range of radii.

- Calculate the Hawking temperature $T(M)$,
  - this is done using (2.11) with r_1 instead of r_0.
- Plot point $(M, T(M))$ for the current mass.
- Add another mass step and repeat.

Working in `Wolfram Mathematica 11.3`, if we use `ListLinePlot` to plot an `Array` of $(M, T(M))$, we can easily sort this array by its second component which will yield us the maximal temperature T_{max} as well as the mass for which this temperature is reached M_1.