Cvičenie 6

21.3.2023

Prepočítané príklady

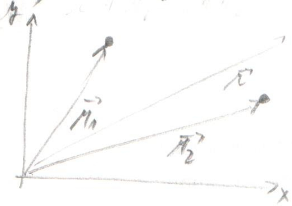
Na cvičení sme rátali príklady 1.3 (odvodenie na konci tohto pdf) a 1.4 a) (riešenie na konci tohto pdf) z časti II. zo skrípt.

Treba si zapamätať

- Poissonova rovnica: $\Delta\varphi = -\frac{\rho}{\varepsilon}$
- Intenzita elektrického poľa: $\vec{E}(\vec{r}) = -\operatorname{grad}\varphi(\vec{r})$
- Plošná hustota náboja:
$$(\partial_n\varphi)|_S = -\frac{\sigma_S}{\varepsilon}$$
- Kapacita vodičov: $Q_i = C_{ij}V_j$

7. cvičenie

1.3 a) 2 náboje, hľadáme $\varphi(\vec{r})=0$



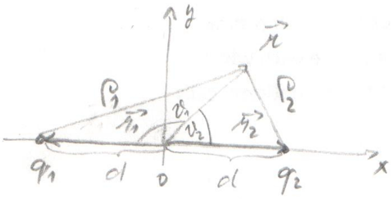
$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{|\vec{r}-\vec{r}_1|} + \frac{q_2}{|\vec{r}-\vec{r}_2|}\right) \qquad |\vec{\rho}_1| = |\vec{r}-\vec{r}_1| \qquad |\vec{\rho}_2| = |\vec{r}-\vec{r}_2|$$

chceme, aby $\varphi(\vec{r})=0 \Rightarrow \frac{q_1}{\rho_1} = \frac{-q_2}{\rho_2} \Rightarrow \frac{-q_1}{q_2} = \frac{\rho_1}{\rho_2} = k > 0$

↳ 1. náboje musia mať opačné znamienka

2. ak $q_1 = -q_2$, $\rho_1 = \rho_2 \Rightarrow$ rovina v strede

↳ 3. ak $q_1 \neq -q_2$, zvolíme vhodnejšiu sústavu



$|\vec{r}_2| = |\vec{r}_1| = d$

$$\left.\begin{aligned} |\vec{\rho}_1| = |\vec{r}-\vec{r}_1| &\Rightarrow \rho_1^2 = r^2 + d^2 - 2rd\cos\vartheta_1 \\ |\vec{\rho}_2| = |\vec{r}-\vec{r}_2| &\Rightarrow \rho_2^2 = r^2 + d^2 - 2rd\cos\vartheta_2 \end{aligned}\right\} \vartheta_1 + \vartheta_2 = \pi$$

$$\rho_1^2 = r^2 + d^2 - 2rd\cos\vartheta_1$$
$$\rho_2^2 = r^2 + d^2 - 2rd\cos(\pi - \vartheta_1) = r^2 + d^2 + 2rd\cos\vartheta_1$$

pričom stále platí $\frac{\rho_1}{\rho_2} = -\frac{q_1}{q_2} \Rightarrow \frac{\rho_1^2}{\rho_2^2} = \frac{q_1^2}{q_2^2} \equiv k \; ; \; k \neq 1$

preto $\rho_1^2 - \frac{q_1^2}{q_2^2}\rho_2^2 = 0$

$$r^2 + d^2 - 2rd\cos\vartheta_1 - \frac{q_1^2}{q_2^2}\left(r^2 + d^2 + 2rd\cos\vartheta_1\right) = 0$$

$$r^2(1-k) + d^2(1-k) - 2rd\cos\vartheta_1(1+k) = 0$$

keďže $r^2 = x^2 + y^2$, tak potom

$$x^2(1-k) + y^2(1-k) + d^2(1-k) - 2rd\cos\vartheta_1(1+k) = 0 \qquad \Big/ \frac{1}{1-k}$$

$$x^2 + y^2 + d^2 - 2rd\cos\vartheta_1 \frac{1+k}{1-k} = 0$$

$$x^2 + y^2 + d^2 - 2rd\cos(\pi - \vartheta_2)\frac{1+k}{1-k} = 0$$

$$x^2 + y^2 + d^2 + 2rd\cos\vartheta_2 \frac{1+k}{1-k} = 0 \qquad \Big/ r\cos\vartheta_2 = x$$

$$x^2 + y^2 + d^2 + 2dx\frac{1+k}{1-k} = 0$$

$$\left(x + d\left(\frac{1+k}{1-k}\right)\right)^2 + y^2 + d^2 - d^2\left(\frac{1+k}{1-k}\right)^2 = 0$$

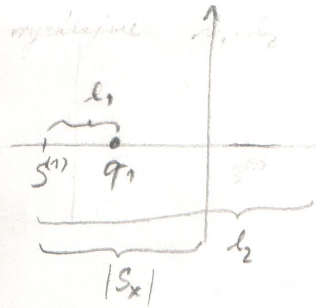
$$\left(x + d\left(\frac{1+k}{1-k}\right)\right)^2 + y^2 = d^2\left(\left(\frac{1+k}{1-k}\right)^2 - 1\right)$$

$\frac{1+k}{1-k} = \frac{1+\frac{q_1^2}{q_2^2}}{1-\frac{q_1^2}{q_2^2}} = \frac{q_1^2+q_2^2}{q_2^2-q_1^2} \quad \Rightarrow S = \left[d\frac{q_1^2+q_2^2}{q_1^2-q_2^2}, 0\right] \quad \Rightarrow |q_1| < |q_2| \ldots$ stred bližšie ku q_1

$\Rightarrow |S_x| > d$ lebo $\left|\frac{q_1^2+q_2^2}{q_1^2-q_2^2}\right| > 1$

$$\left(\frac{1+k}{1-k}\right)^2 - 1 = \frac{(q_1^2+q_2^2)^2 - (q_2^2-q_1^2)^2}{(q_2^2-q_1^2)^2} = \frac{\cancel{q_1^4} + 2q_1^2q_2^2 + \cancel{q_2^4} - \cancel{q_2^4} + 2q_1^2q_2^2 - \cancel{q_1^4}}{(q_2^2-q_1^2)^2} = \frac{4q_1^2q_2^2}{(q_2^2-q_1^2)^2}$$

$$\Rightarrow R = d\,\frac{2|q_1||q_2|}{|q_2^2-q_1^2|}$$



(1) $|q_1| < |q_2| \quad \dots \quad S_x < 0$

$$l_1 = -S_x - d \qquad\qquad l_2 = -S_x + d$$

(2) $|q_1| > |q_2| \quad \dots \quad S_x > 0$

$$l_1 = d + S_x \qquad l_2 = S_x - d$$

(1),(2) $l_1 \cdot l_2 = S_x^2 - d^2 = d^2\left(\frac{1+k}{1-k}\right)^2 - d^2 = d^2\left(\left(\frac{1+k}{1-k}\right)^2 - 1\right) = R^2$

$$\frac{R}{l_2}|q_2| \overset{?}{=} |q_1| \to \frac{R^2}{l_2^2} q_2^2 \overset{(1)}{=} \frac{R^2}{S_x^2 - 2dS_x + d^2} q_2^2 = \frac{d^2\left(\left(\frac{1+k}{1-k}\right)^2 - 1\right)}{d^2\left(\frac{1+k}{1-k}\right)^2 + 2d^2\left(\frac{1+k}{1-k}\right) + d^2} q_2^2 =$$

$$= \frac{\alpha^2 - 1}{\alpha^2 + 2\alpha + 1} q_2^2 = \frac{(\alpha - 1)\cancel{(\alpha+1)}}{(\alpha+1)\cancel{(\alpha+1)}} q_2^2 = \frac{\frac{1+k}{1-k} - 1}{\frac{1+k}{1-k} + 1} q_2^2 =$$

$$= \frac{1+k-1+k}{1+k+1-k} q_2^2 = \frac{2k}{2} q_2^2 = k q_2^2 = \frac{q_1^2}{q_2^2} q_2^2 = q_1^2$$

$$\Rightarrow \frac{R^2}{l_2^2} q_2^2 = q_1^2 \Rightarrow \frac{R}{l_2}|q_2| = |q_1| \Rightarrow \frac{R}{l_2} q_2 = -q_1$$

$\Rightarrow$ zhrnutie: kružnica $\to$ $S = \left[d\,\frac{q_1^2 + q_2^2}{q_1^2 - q_2^2}, 0\right]$ (keď zvolíme počiatok v (0,0))

↳ stred kružnice je bližšie pri menšom náboji

↳ $|S_x| > d$ $\to$ stred kružnice nikdy nie je medzi nábojmi

$\to$ $R = d\,\frac{2|q_1||q_2|}{|q_2^2 - q_1^2|}$ $\to$ je vždy taký, že vnútri je len jeden náboj

$$\to \frac{R}{l_2} q_2 = -q_1 \quad \wedge \quad l_1 l_2 = R^2$$

c) sférická ekvipotenciálna hladina s potenciálom V

↳ zoberiem pôvodné, čo vytvárajú sférickú plochu s nulovým potenciálom, k nim pridám ďalší náboj do stredu sféry, potom

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$$