Nech $\vec{a}$ a $\vec{b}$ sú ľubovoľné konštantné vektory a $\vec{r} \equiv (x, y, z) \equiv (x_1, x_2, x_3)$ je polohový vektor v kartézskej súradnicovej sústave. Vypočítať:

1. div $\vec{r}$,
2. rot $\vec{r}$,
3. grad $(\vec{a} \cdot \vec{r})$,
4. div $[(\vec{a} \cdot \vec{r})\,\vec{r}]$,
5. rot $[(\vec{a} \cdot \vec{r})\,\vec{r}]$,
6. div $(\vec{a} \times \vec{r})$,
7. rot $(\vec{a} \times \vec{r})$,
8. grad $\dfrac{1}{r}$,
9. grad $\left[(\vec{a} \cdot \vec{r})\left(\vec{b} \cdot \vec{r}\right)\right]$,
10. grad $\left[\dfrac{(\vec{a} \cdot \vec{r})}{r}\right]$,
11. grad $[\ln(\vec{a} \cdot \vec{r})]$,
12. div $\left(\dfrac{\vec{r}}{r}\right)$,
13. div $\{[\ln(\vec{a} \cdot \vec{r})]\,\vec{r}\}$,
14. rot $\left[\dfrac{(\vec{a} \times \vec{r})}{r}\right]$,
15. rot $[\vec{r} \times (\vec{a} \times \vec{r})]$,
16. grad $\left[\dfrac{(\vec{a} \cdot \vec{r})}{r^3}\right]$,
17. div $\left[\dfrac{(\vec{a} \cdot \vec{r})\left(\vec{b} \times \vec{r}\right)}{r^3}\right]$,
18. rot $\left[\dfrac{(\vec{a} \times \vec{r})}{r^3}\right]$,
19. rot $[\vec{r} - (\vec{a} \cdot \vec{r})\,\vec{a}]$,
20. rot $[\vec{r} - (\vec{a} \cdot \vec{r})\,\vec{r}]$,
21. div $[\vec{r} - (\vec{a} \cdot \vec{r})\,\vec{a}]$,
22. div $[\vec{r} - (\vec{a} \cdot \vec{r})\,\vec{r}]$,
23. grad $\left\{[\vec{r} - (\vec{a} \cdot \vec{r})\,\vec{r}]^2\right\}$,
24. grad $\left\{[\vec{r} - (\vec{a} \cdot \vec{r})\,\vec{a}]^2\right\}$.

Výsledky sú na ďalšej strane.

1. $\operatorname{div} \vec{r} = 3$,
2. $\operatorname{rot} \vec{r} = 0$,
3. $\operatorname{grad} \ (\vec{a} \cdot \vec{r}) = \vec{a}$,
4. $\operatorname{div} \ [(\vec{a} \cdot \vec{r})\, \vec{r}] = 4\vec{a} \cdot \vec{r}$,
5. $\operatorname{rot} \ [(\vec{a} \cdot \vec{r})\, \vec{r}] = \vec{a} \times \vec{r}$,
6. $\operatorname{div} \ (\vec{a} \times \vec{r}) = 0$,
7. $\operatorname{rot} \ (\vec{a} \times \vec{r}) = 2\vec{a}$,
8. $\operatorname{grad} \dfrac{1}{r} = -\dfrac{1}{r^3}\vec{r}$,
9. $\operatorname{grad} \ \left[(\vec{a} \cdot \vec{r}) \left(\vec{b} \cdot \vec{r}\right)\right] = (\vec{a} \cdot \vec{r})\, \vec{b} + \left(\vec{b} \cdot \vec{r}\right) \vec{a}$,
10. $\operatorname{grad} \ \left[\dfrac{(\vec{a} \cdot \vec{r})}{r}\right] = -\dfrac{1}{r^3} (\vec{a} \cdot \vec{r})\, \vec{r} + \dfrac{1}{r}\vec{a}$,
11. $\operatorname{grad} \ [\ln (\vec{a} \cdot \vec{r})] = \dfrac{1}{(\vec{a} \cdot \vec{r})}\vec{a}$,
12. $\operatorname{div} \ \left(\dfrac{\vec{r}}{r}\right) = 2\dfrac{1}{r}$,
13. $\operatorname{div} \ \{[\ln (\vec{a} \cdot \vec{r})]\, \vec{r}\} = 1 + 3 \ln (\vec{a} \cdot \vec{r})$,
14. $\operatorname{rot} \ \left[\dfrac{(\vec{a} \times \vec{r})}{r}\right] = \dfrac{1}{r^3} (\vec{a} \cdot \vec{r})\, \vec{r} + \dfrac{1}{r}\vec{a}$,
15. $\operatorname{rot} \ [\vec{r} \times (\vec{a} \times \vec{r})] = -3\vec{a} \times \vec{r}$,
16. $\operatorname{grad} \ \left[\dfrac{(\vec{a} \cdot \vec{r})}{r^3}\right] = -\dfrac{3}{r^5} (\vec{a} \cdot \vec{r})\, \vec{r} + \dfrac{1}{r^3}\vec{a}$,
17. $\operatorname{div} \ \left[\dfrac{(\vec{a} \cdot \vec{r}) \left(\vec{b} \times \vec{r}\right)}{r^3}\right] = \dfrac{\vec{a} \cdot \left(\vec{b} \times \vec{r}\right)}{r^3}$,
18. $\operatorname{rot} \ \left[\dfrac{(\vec{a} \times \vec{r})}{r^3}\right] = \dfrac{3\, (\vec{a} \cdot \vec{r})}{r^5}\vec{r} - \dfrac{1}{r^3}\vec{a}$,
19. $\operatorname{rot} \ [\vec{r} - (\vec{a} \cdot \vec{r})\, \vec{a}] = 0$,
20. $\operatorname{rot} \ [\vec{r} - (\vec{a} \cdot \vec{r})\, \vec{r}] = -\vec{a} \times \vec{r}$,
21. $\operatorname{div} \ [\vec{r} - (\vec{a} \cdot \vec{r})\, \vec{a}] = 3 - \vec{a}^2$,
22. $\operatorname{div} \ [\vec{r} - (\vec{a} \cdot \vec{r})\, \vec{r}] = 3 - 4\vec{a} \cdot \vec{r}$,
23. $\operatorname{grad} \ \left\{[\vec{r} - (\vec{a} \cdot \vec{r})\, \vec{r}]^2\right\} = 2\,(1 - \vec{a} \cdot \vec{r})\, [(1 - \vec{a} \cdot \vec{r})\, \vec{r} - r^2\vec{a}]$,
24. $\operatorname{grad} \ \left\{[\vec{r} - (\vec{a} \cdot \vec{r})\, \vec{a}]^2\right\} = 2\, [\vec{r} + (\vec{a}^2 - 2)\, (\vec{a} \cdot \vec{r})\, \vec{a}]$.

Riešenia sú na ďalších stranách.

PREREKVIZITA $\quad \mathrm{grad}\, r$

$$\partial_i r = \partial_i \sqrt{x_j x_j} = \frac{1}{2}(x_j x_j)^{-\frac{1}{2}} \partial_i x_k x_k =$$

$$= \frac{1}{2r}(\delta_{ik} x_k + x_k \delta_{ik}) = \frac{1}{2r}(x_i + x_i) = \frac{x_i}{r}$$

① $\mathrm{div}\,\vec{r} = \partial_i x_i = \delta_{ii} = \sum_{i=1}^{3} 1 = 3$

② $(\mathrm{rot}\,\vec{r})_i = \varepsilon_{ijk}\,\partial_j x_k = \varepsilon_{ijk}\,\delta_{jk} = \varepsilon_{i[jk]}\,\delta_{(jk)} \overset{*}{=} 0$

$$* \; A_{ij} S_{ij} = A_{\underset{(-1)}{[ij]}} S_{\underset{(+1)}{(ij)}} = -A_{ji} S_{ji} =$$

PREMENOVANIE INDEXOV

$$= -A_{mn} S_{mn} = -A_{ij} S_{ij} = 0$$

NIEČO = − TO ISTÉ

③ $[\mathrm{grad}\,(\vec{a}\cdot\vec{r})]_i = \partial_i a_j x_j = a_j \partial_i x_j = a_j \delta_{ij} = a_i$

ALTERNATÍVNE ZNAČENIE $\left(= (a_j x_j)_{,i} = a_j x_{j,i} = \right)$

④ $\mathrm{div}\,[(\vec{a}\cdot\vec{r})\vec{r}] = \partial_i \overbrace{a_k x_k}^{\vec{a}\cdot\vec{r}} x_i = a_k (x_k x_i)_{,i} =$

$$= a_k (x_{k,i} x_i + x_k x_{i,i}) = a_k (\delta_{ki} x_i + x_k \delta_{ii}) =$$

$$= a_k (x_k + x_k 3) = 4 a_k x_k = 4 \vec{a}\cdot\vec{r}$$

⑤ $\left(\mathrm{rot}\,[(\vec{a}\cdot\vec{r})\vec{r}]\right)_i = \varepsilon_{ijk}\,\partial_j [(\vec{a}\cdot\vec{r})\vec{r}]_k =$

$$= \varepsilon_{ijk}\,\partial_j a_m x_m x_k = \varepsilon_{ijk}\, a_m (x_m x_k)_{,j} =$$

$$= \varepsilon_{ijk}\, a_m (x_{m,j} x_k + x_m x_{k,j}) =$$

$$= \varepsilon_{ijk}\, a_m \delta_{mj} x_k + \varepsilon_{ijk}\, a_m x_m \delta_{kj} =$$

$$= \varepsilon_{ijk}\, a_j x_k + (\vec{a}\cdot\vec{r}) \underbrace{\varepsilon_{ijk}\,\delta_{kj}}_{0\;*} =$$

$$= (\vec{a}\times\vec{r})_i$$

(6) $\operatorname{div}(\vec a\times\vec r) = \partial_i(\vec a\times\vec r)_i = \partial_i\, \varepsilon_{ijk}\, a_j x_k =$

$$= \varepsilon_{ijk}\, a_j\, \partial_i x_k = \underset{\text{ANTISYM}}{\varepsilon_{\underline{i}j\underline{k}}}\, a_j\, \underset{\text{SYM.}}{\underline{\delta_{ik}}} \overset{*}{=} 0$$

(7) $[\operatorname{rot}(\vec a\times\vec r)]_i = \varepsilon_{ijk}\,\partial_j(\vec a\times\vec r)_k = \varepsilon_{ijk}\,\partial_j\,\varepsilon_{kmn}\,a_m x_n =$

$$= \underbrace{\varepsilon_{ijk}\,\varepsilon_{kmn}}_{\text{"DAVIS CUP"}}\, a_m\, \underbrace{\partial_j x_n}_{\delta_{jn}} = \delta_{im}\delta_{jn}\,a_m\,\delta_{jn} - \delta_{in}\delta_{jm}\,a_m\,\delta_{jn} =$$

$$\varepsilon_{kij}\,\varepsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

$$= \delta_{jn}\,a_i\,\delta_{jn} - \delta_{jm}\,a_m\,\delta_{ji} =$$

$$= a_i\,\delta_{jj} - a_j\,\delta_{ji} =$$

$$= a_i 3 - a_i = 2a_i$$

(8) $\left(\operatorname{grad}\frac{1}{r}\right)_i = \partial_i\frac{1}{r} = -\frac{1}{r^2}\partial_i r = -\frac{1}{r^2}\frac{x_i}{r} = \left(-\frac{1}{r^3}\vec r\right)_i$

(9) $\left(\operatorname{grad}[(\vec a\cdot\vec r)(\vec b\cdot\vec r)]\right)_i = \partial_i\, \overbrace{a_j x_j}^{\vec a\cdot\vec r}\, \overbrace{b_k x_k}^{\vec b\cdot\vec r} = a_j b_k\, \partial_i x_j x_k =$

$$= a_j b_k (x_{j,i} x_k + x_j x_{k,i}) = a_j b_k\,\delta_{ji} x_k + a_j b_k x_j\,\delta_{ki} =$$

$$= a_i b_k x_k + a_j b_i x_j = a_i(\vec b\cdot\vec r) + (\vec a\cdot\vec r) b_i =$$

$$= [(\vec a\cdot\vec r)\vec b + (\vec b\cdot\vec r)\vec a]_i$$

(10) $\left(\operatorname{grad}\left[\frac{(\vec a\cdot\vec r)}{r}\right]\right)_i = \partial_i\left(\frac{a_j x_j}{r}\right) = -\frac{1}{r^2} r_{,i}\, a_j x_j + \frac{a_j}{r}\partial_i x_j =$

$$= -\frac{1}{r^2}\frac{x_i}{r} a_j x_j + \frac{a_j}{r}\delta_{ij} = -\frac{1}{r^3} x_i(\vec a\cdot\vec r) + \frac{a_i}{r} =$$

$$= \left(-\frac{1}{r^3}(\vec a\cdot\vec r)\vec r + \frac{1}{r}\vec a\right)_i$$

(11) $\left(\operatorname{grad}[\ln(\vec a\cdot\vec r)]\right)_i = \partial_i \ln(a_j x_j) = \frac{1}{a_j x_j}\partial_i a_k x_k =$

$$= \frac{1}{\vec a\cdot\vec r} a_k\,\partial_i x_k = \frac{1}{\vec a\cdot\vec r} a_k\,\delta_{ik} = \frac{1}{\vec a\cdot\vec r} a_i = \left(\frac{1}{\vec a\cdot\vec r}\vec a\right)_i$$

(12) $\operatorname{div}\left(\frac{\vec r}{r}\right) = \partial_i\frac{x_i}{r} = -\frac{1}{r^2}(\partial_i r)x_i + \frac{1}{r}\partial_i x_i = -\frac{1}{r^2}\frac{x_i}{r}x_i + \frac{1}{r}\delta_{ii} =$

$$= -\frac{1}{r^3}\underbrace{x_i x_i}_{r^2} + \frac{1}{r}3 = -\frac{1}{r} + 3\frac{1}{r} = 2\frac{1}{r}$$

(13) $\operatorname{div}\left\{\left[\ln(\vec{a}\cdot\vec{r})\right]\vec{r}\right\} = \partial_i\left\{\left[\ln(a_j x_j)\right]x_i\right\} =$

$$= \frac{1}{a_j x_j}\underbrace{\left(\partial_i a_k x_k\right)}_{a_k \partial_i x_k} x_i + \ln(a_j x_j)\,\partial_i x_i = \frac{1}{a_j x_j}\, a_k \delta_{ik} x_i + \ln(a_j x_j)\,\delta_{ii} =$$

$$= \frac{1}{a_j x_j}\, a_i x_i + 3\ln(a_i x_i) = 1 + 3\ln(\vec{a}\cdot\vec{r})$$

(14) $\left(\operatorname{rot}\frac{\vec{a}\times\vec{r}}{r}\right)_i = \varepsilon_{ijk}\,\partial_j \frac{(\vec{a}\times\vec{r})_k}{r} = \varepsilon_{ijk}\,\partial_j \frac{\varepsilon_{kmn} a_m x_n}{r} =$

$$= \underset{\text{CYKL.}}{\varepsilon_{ijk}}\,\frac{-1}{r^2}(\partial_j r)\,\varepsilon_{kmn} a_m x_n + \underset{\text{CYKL.}}{\varepsilon_{ijk}}\,\frac{\varepsilon_{kmn} a_m}{r}\,\partial_j x_n =$$

$$= \underline{\varepsilon_{kij}}\,\frac{-1}{r^2}\,\frac{x_j}{r}\,\underline{\varepsilon_{kmn}}\, a_m x_n + \underline{\varepsilon_{kij}\,\varepsilon_{kmn}}\,\frac{a_m}{r}\,\delta_{jn} =$$

$$= \Big(\overbrace{\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}}^{\varepsilon_{kij}\varepsilon_{kmn}}\Big)\left(-\frac{1}{r^3}x_j a_m x_n + \frac{1}{r}a_m \delta_{jn}\right) =$$

$$= -\frac{1}{r^3}\delta_{im}\delta_{jn} x_j a_m x_n + \frac{1}{r}\delta_{im}\delta_{jn} a_m \delta_{jn} +$$

$$+ \frac{1}{r^3}\delta_{in}\delta_{jm} x_j a_m x_n - \frac{1}{r}\delta_{in}\delta_{jm} a_m \delta_{jn} =$$

$$= -\frac{1}{r^3}\delta_{jn} x_j a_i x_n + \frac{1}{r}\delta_{jn} a_i \delta_{jn} +$$

$$+ \frac{1}{r^3}\delta_{jm} x_j a_m x_i - \frac{1}{r}\delta_{jm} a_m \delta_{ji} =$$

$$= -\frac{1}{r^3}\underbrace{x_j a_i x_j}_{r^2 a_i} + \frac{1}{r}\underbrace{a_i \delta_{jj}}_{3a_i} + \frac{1}{r^3}\underbrace{x_j a_j x_i}_{(\vec{a}\cdot\vec{r})x_i} - \frac{1}{r}\underbrace{a_j \delta_{ji}}_{a_i} =$$

$$= \left(-\frac{1}{r}\vec{a} + \frac{1}{r}3\vec{a} + \frac{1}{r^3}(\vec{a}\cdot\vec{r})\vec{r} - \frac{1}{r}\vec{a}\right)_i =$$

$$= \left(\frac{1}{r^3}(\vec{a}\cdot\vec{r})\vec{r} + \frac{1}{r}\vec{a}\right)_i$$

(15) $$\left(\mathrm{rot}\,[\vec r\times(\vec a\times\vec r)]\right)_i = \varepsilon_{ijk}\partial_j[\vec r\times(\vec a\times\vec r)]_k =$$

$$= \varepsilon_{ijk}\partial_j\varepsilon_{kmn}x_m(\vec a\times\vec r)_n =$$

$$= \varepsilon_{ijk}\partial_j\varepsilon_{kmn}x_m\varepsilon_{nab}a_a x_b =$$

$$= \varepsilon_{ijk}\overset{\text{cykl}}{\varepsilon_{kmn}}(\partial_j x_m)\varepsilon_{nab}a_a x_b +$$

$$+ \varepsilon_{ijk}\varepsilon_{kmn}x_m\varepsilon_{nab}a_a(\partial_j x_b) =$$

$$= \underbrace{\varepsilon_{ijk}\varepsilon_{mnk}}_{\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}}\varepsilon_{nab}a_a(\delta_{jm}x_b + x_m\delta_{jb}) =$$

$$= \delta_{im}\delta_{jn}\varepsilon_{nab}a_a\delta_{jm}x_b + \delta_{im}\delta_{jn}\varepsilon_{nab}a_a x_m\delta_{jb} -$$

$$- \delta_{in}\delta_{jm}\varepsilon_{nab}a_a\delta_{jm}x_b - \delta_{in}\delta_{jm}\varepsilon_{nab}a_a x_m\delta_{jb} =$$

$$= \delta_{jn}\varepsilon_{nab}a_a\delta_{ji}x_b + \delta_{jn}\varepsilon_{nab}a_a x_i\delta_{jb} -$$

$$- \delta_{jm}\varepsilon_{iab}a_a\delta_{jm}x_b - \delta_{jm}\varepsilon_{iab}a_a x_m\delta_{jb} =$$

$$= \varepsilon_{jab}a_a\delta_{ji}x_b + \varepsilon_{jab}x_i\delta_{jb} -$$

$$- \varepsilon_{iab}a_a\delta_{jj}x_b - \varepsilon_{iab}a_a x_j\delta_{jb} =$$

$$= \varepsilon_{iab}a_a x_b + \underbrace{\varepsilon_{bab}}_{0}x_i - 3\varepsilon_{iab}a_a x_b - \varepsilon_{iaj}a_a x_j =$$

$$= \left(\vec a\times\vec r - 3\,\vec a\times\vec r - \vec a\times\vec r\right)_i = \left(-3\,\vec a\times\vec r\right)_i$$

(16) $$\left(\mathrm{grad}\,\frac{\vec a\cdot\vec r}{r^3}\right)_i = \partial_i\left(\frac{a_j x_j}{r^3}\right) = \frac{-3}{r^4}(\partial_i r)a_j x_j + \frac{a_j}{r^3}\partial_i x_j =$$

$$= -\frac{3}{r^4}\frac{x_i}{r}a_j x_j + \frac{a_j}{r^3}\delta_{ij} = \left(-\frac{3}{r^5}(\vec a\cdot\vec r)\vec r + \frac{1}{r^3}\vec a\right)_i$$

17. $$\operatorname{div}\left[\frac{(\vec a\cdot\vec r)(\vec b\times\vec r)}{r^3}\right] = \partial_i \frac{(\vec a\cdot\vec r)(\vec b\times\vec r)_i}{r^3} =$$

$$= \partial_i \frac{a_m x_m \varepsilon_{ijk} b_j x_k}{r^3} =$$

$$= -\frac{3}{r^4}(\partial_i r)\, a_m x_m \varepsilon_{ijk} b_j x_k + \frac{a_m(\partial_i x_m)\varepsilon_{ijk} b_j x_k}{r^3} +$$

$$+ \frac{a_m x_m \varepsilon_{ijk} b_j \partial_i x_k}{r^3} = \underbrace{-\frac{3}{r^4}\frac{x_i}{r}\, a_m x_m \varepsilon_{ijk} b_j x_k}_{=0} +$$

$$x_i \varepsilon_{ijk} x_k = -\varepsilon_{j\underset{\text{ANTISYM}}{ik}}\, \underset{\text{SYM}}{x_i x_k} = 0 = \varepsilon_{\underset{\text{ANTISYM}}{ijk}}\, \underset{\text{SYM}}{\delta_{ik}}$$

$$+ \frac{a_m \delta_{im} \varepsilon_{ijk} b_j x_k}{r^3} + \overbrace{\frac{a_m x_m \varepsilon_{ijk} b_j \delta_{ik}}{r^3}}^{=0} =$$

$$= \frac{a_i \varepsilon_{ijk} b_j x_k}{r^3} = \frac{\vec a\cdot(\vec b\times\vec r)}{r^3}$$

18. $$\left(\operatorname{rot}\left(\frac{\vec a\times\vec r}{r^3}\right)\right)_i = \varepsilon_{ijk}\partial_j \frac{(\vec a\times\vec r)_k}{r^3} =$$

$$= \varepsilon_{ijk}\partial_j \frac{\varepsilon_{kmn} a_m x_n}{r^3} = \varepsilon_{ijk}\frac{-3}{r^4}(\partial_j r)\,\varepsilon_{kmn} a_m x_n +$$

$$+ \varepsilon_{ijk}\varepsilon_{kmn} a_m \frac{1}{r^3}\partial_j x_n = \underset{\text{CYKL}}{\varepsilon_{ijk}} \frac{-3}{r^4}\frac{x_j}{r}\varepsilon_{kmn} a_m x_n +$$

$$+ \underset{\text{CYKL}}{\varepsilon_{ijk}}\varepsilon_{kmn} a_m \frac{1}{r^3}\delta_{jn} = \varepsilon_{kij}\varepsilon_{kmn}\left(-\frac{3}{r^5}x_j a_m x_n +\right.$$

$$\left.+ \frac{1}{r^3} a_m \delta_{jn}\right) = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\left(-\frac{3}{r^5}x_j a_m x_n + \frac{1}{r^3}a_m\delta_{jn}\right) =$$

$$= \delta_{im}\delta_{jn}\frac{-3}{r^5}x_j a_m x_n + \delta_{im}\delta_{jn}\frac{1}{r^3}a_m\delta_{jn} -$$

$$- \delta_{in}\delta_{jm}\frac{-3}{r^5}x_j a_m x_n - \delta_{in}\delta_{jm}\frac{1}{r^3}a_m\delta_{jn} =$$

$$= \delta_{jn}\frac{-3}{r^5}x_j a_i x_n + \delta_{jn}\frac{1}{r^3}a_i\delta_{jn} - \delta_{jm}\frac{-3}{r^5}x_j a_m x_i - \delta_{jm}\frac{1}{r^3}a_m\delta_{ji} =$$

$$= -\frac{3}{r^5}x_j a_i x_j + \frac{1}{r^3}a_i\delta_{jj} + \frac{3}{r^5}x_j a_j x_i - \frac{1}{r^3}a_j\delta_{ji} =$$

$$= \left(-\frac{3}{r^5}r^2\vec a + \frac{1}{r^3}3\vec a + \frac{3}{r^5}(\vec a\cdot\vec r)\vec r - \frac{1}{r^3}\vec a\right)_i = \left(\frac{3(\vec a\cdot\vec r)}{r^5}\vec r - \frac{\vec a}{r^3}\right)_i$$

(19) $\left(\mathrm{rot}\left[\vec{r}-(\vec{a}\cdot\vec{r})\vec{a}\right]\right)_i = \varepsilon_{ijk}\partial_j\left[\vec{r}-(\vec{a}\cdot\vec{r})\vec{a}\right]_k =$

$= \varepsilon_{ijk}\partial_j\left(x_k - a_m x_m a_k\right) = \varepsilon_{ijk}\left(\partial_j x_k - a_m(\partial_j x_m)a_k\right) =$

$= \underbrace{\underset{\text{ANTISYM}}{\varepsilon_{ijk}}\,\underset{\text{SYM}}{\delta_{jk}}}_{0} - \varepsilon_{ijk}a_m\delta_{jm}a_k = -\underset{\text{ANTISYM}}{\varepsilon_{ijk}}\,\underset{\text{SYM}}{a_j a_k} = 0$

$(\vec{a}\times\vec{a}=0)$

(20) $\left(\mathrm{rot}\left[\vec{r}-(\vec{a}\cdot\vec{r})\vec{r}\right]\right)_i = \varepsilon_{ijk}\partial_j\left[\vec{r}-(\vec{a}\cdot\vec{r})\vec{r}\right]_k =$

$= \varepsilon_{ijk}\partial_j\left(x_k - a_m x_m x_k\right) = \varepsilon_{ijk}\left(\partial_j x_k - a_m(\partial_j x_m)x_k\right) =$

$= \varepsilon_{ijk}\left(\delta_{jk} - a_m\delta_{jm}x_k\right) = \underset{\text{ANTISYM}}{\varepsilon_{ijk}}\left(\underset{\text{SYM}}{\delta_{jk}} - a_j x_k\right) =$

$= -\varepsilon_{ijk}a_j x_k = \left(-\vec{a}\times\vec{r}\right)_i$

(21) $\mathrm{div}\left[\vec{r}-(\vec{a}\cdot\vec{r})\vec{a}\right] = \partial_i\left[\vec{r}-(\vec{a}\cdot\vec{r})\vec{a}\right]_i =$

$= \partial_i\left(x_i - a_j x_j a_i\right) = \partial_i x_i - a_j(\partial_i x_j)a_i =$

$= \delta_{ii} - a_j\delta_{ij}a_i = 3 - a_j a_j = 3 - \vec{a}^2$

(22) $\mathrm{div}\left[\vec{r}-(\vec{a}\cdot\vec{r})\vec{r}\right] = \partial_i\left(x_i - a_j x_j x_i\right) = \partial_i x_i - a_j(\partial_i x_j)x_i -$

$- a_j x_j(\partial_i x_i) = \delta_{ii} - a_j\delta_{ij}x_i - a_j x_j\delta_{ii} =$

$= 3 - a_j x_j - a_j x_j 3 = 3 - 4a_j x_j = 3 - 4\,\vec{a}\cdot\vec{r}$

23 $\left(\text{grad}\left\{[\vec{r}-(\vec{a}\cdot\vec{r})\vec{r}]^2\right\}\right)_i = \partial_i\left([\vec{r}-(\vec{a}\cdot\vec{r})\vec{r}]^2\right) =$

$= \partial_i\, [\vec{r}-(\vec{a}\cdot\vec{r})\vec{r}]_j\, [\vec{r}-(\vec{a}\cdot\vec{r})\vec{r}]_j =$

$= \partial_i\left[(x_j - a_k x_k x_j)(x_j - a_m x_m x_j)\right] =$

$= \partial_i\left(x_j x_j - x_j a_m x_m x_j - a_k x_k x_j x_j + a_k x_k a_m x_m x_j x_j\right) =$

$= x_{j,i} x_j + x_j x_{j,i} - x_{j,i} a_m x_m x_j - x_j a_m x_{m,i} x_j - x_j a_m x_m x_{j,i} -$

$- a_k x_{k,i} x_j x_j - a_k x_k x_{j,i} x_j - a_k x_k x_j x_{j,i} +$

$+ a_k x_{k,i} a_m x_m x_j x_j + a_k x_k a_m x_{m,i} x_j x_j +$

$+ a_k x_k a_m x_m x_{j,i} x_j + a_k x_k a_m x_m x_j x_{j,i} =$

$= 2\,\delta_{ji} x_j - \delta_{ji} a_m x_m x_j - x_j a_m \delta_{mi} x_j - x_j a_m x_m \delta_{ji} -$

$- a_k \delta_{ki} x_j x_j - 2\, a_k x_k \delta_{ji} x_j +$

$+ a_k \delta_{ki} a_m x_m x_j x_j + a_k x_k a_m \delta_{mi} x_j x_j +$

$+ 2\, a_k x_k a_m x_m \delta_{ji} x_j =$

$= 2 x_i - (\vec{a}\cdot\vec{r}) x_i - r^2 a_i - (\vec{a}\cdot\vec{r}) x_i -$

$- r^2 a_i - 2(\vec{a}\cdot\vec{r}) x_i +$

$+ a_i (\vec{a}\cdot\vec{r}) r^2 + (\vec{a}\cdot\vec{r}) a_i r^2 +$

$+ 2(\vec{a}\cdot\vec{r})^2 x_i =$

$= \left[\left(2 - 4\,\vec{a}\cdot\vec{r} + 2(\vec{a}\cdot\vec{r})^2\right)\vec{r} + \left(-2r^2 + 2(\vec{a}\cdot\vec{r})r^2\right)\vec{a}\right]_i =$

$= \left[2(1-\vec{a}\cdot\vec{r})^2\vec{r} - 2r^2(1-\vec{a}\cdot\vec{r})\vec{a}\right]_i =$

$= \left\{2(1-\vec{a}\cdot\vec{r})\left[(1-\vec{a}\cdot\vec{r})\vec{r} - r^2\vec{a}\right]\right\}_i$

(24) $\left(\mathrm{grad}\left\{ \left[\vec{r} - (\vec{a}\cdot\vec{r})\vec{a} \right]^2 \right\} \right)_i = \partial_i \left(\left[\vec{r} - (\vec{a}\cdot\vec{r})\vec{a} \right]^2 \right) =$

$= \partial_i \left[\vec{r} - (\vec{a}\cdot\vec{r})\vec{a} \right]_j \left[\vec{r} - (\vec{a}\cdot\vec{r})\vec{a} \right]_j =$

$= \partial_i \left[(x_j - a_m x_m a_j)(x_j - a_k x_k a_j) \right] =$

$= \partial_i \left(x_j x_j - x_j a_k x_k a_j - a_m x_m a_j x_j + a_m x_m a_k x_k a_j a_j \right) =$

$= x_{j,i} x_j + x_j x_{j,i} - x_{j,i} a_k x_k a_j - x_j a_k x_{k,i} a_j -$

$- a_m x_{m,i} a_j x_j - a_m x_m a_j x_{j,i} +$

$+ a_m x_{m,i} a_k x_k a_j a_j + a_m x_m a_k x_{k,i} a_j a_j =$

$= 2 \delta_{ji} x_j - \delta_{ji} a_k x_k a_j - x_j a_k \delta_{ki} a_j -$

$- a_m \delta_{mi} a_j x_j - a_m x_m a_j \delta_{ji} +$

$+ a_m \delta_{mi} a_k x_k a_j a_j + a_m x_m a_k \delta_{ki} a_j a_j =$

$= 2x_i - (\vec{a}\cdot\vec{r}) a_i - (\vec{a}\cdot\vec{r}) a_i -$

$- a_i (\vec{a}\cdot\vec{r}) - (\vec{a}\cdot\vec{r}) a_i +$

$+ a_i (\vec{a}\cdot\vec{r}) \vec{a}^2 + (\vec{a}\cdot\vec{r}) a_i \vec{a}^2 =$

$= \left(2\vec{r} - 4(\vec{a}\cdot\vec{r})\vec{a} + 2(\vec{a}\cdot\vec{r})(\vec{a}^2)\vec{a} \right)_i =$

$= \left\{ 2\left[\vec{r} + (\vec{a}^2 - 2)(\vec{a}\cdot\vec{r})\vec{a} \right] \right\}_i$