

Quantum spacetime and gravity from the IKKT matrix model: an invitation

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Abstract

This is a **preliminary version** of a concise & compact introduction to a framework for spacetime, gravity and fundamental physics based on the IKKT matrix model, focusing on conceptual and structural aspects. The approach rests on non-trivial vacua or sufficiently stable matrix backgrounds, which serve as 3+1-dimensional noncommutative spacetime. A specific background describing an expanding homogeneous and isotropic FLRW spacetime is described, which is a solution of the one-loop effective action with finitely many dof per volume. The matrix fluctuations lead to fields propagating on spacetime governed by a weakly-coupled gauge theory, and a non-standard description of (quantum) gravity leading to some IR modifications. Maximal supersymmetry of the model is essential to ensure effective locality.

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1 Introduction

Nature is governed by quantum mechanics. This is the most profound insight of 20th century physics, replacing a classical picture that seemed satisfactory but is ultimately inconsistent. Today, we still face an analogous problem with gravity: general relativity (GR), its currently accepted description, is a classical theory that is ultimately inconsistent and bound to fail, due to inevitable singularities of spacetime. In a consistent theory, matter falling into a black hole ought to be described and accommodated in some meaningful way. This is analogous to the classical instability of atoms, which signals the breakdown of classical physics. Resolving these issues will likely require to go beyond the classical framework of geometry.

The problem of spacetime singularities – as well as other issues such as black hole entropy – suggests that spacetime admits only finitely many degrees of freedom per volume. In the context of a renormalizable quantum theory, such a discretization is easily accommodated, and its specific implementation – e.g. through a lattice-formulation – would be irrelevant at

low energies. However the non-renormalizability of general relativity implies that different discretized versions of GR (such as loop quantum gravity, dynamical triangulations, and many other proposals) would lead to totally different physics, and are basically ad-hoc. There might be preferred versions as suggested by asymptotic safety [1, 2], but a compelling theory should have a clear and simple mathematical definition.

This issue is of course addressed in string theory, which starts from an entirely different and "special" framework that provides some well-behaved quantum theory, aiming to obtain an effective quantum theory of 3+1D gravity at low energies. That however leads to another issue: string theory lives in 9+1 dimensions, and standard methods to reduce it down to 3+1 dimensions introduce vast amounts of ambiguities. This issue is known as the landscape, and seems hard to accept for a fundamental theory.

The purpose of this paper is to discuss the foundations of a different approach to quantum gravity, based on a distinct matrix model known as IKKT or IIB model with action of the form $S = \text{tr}([\mathbf{T}, \mathbf{T}]^2 + \dots)$ [3]. While this model belongs to the broad arena of string theory and inherits much of its magic, it provides a non-perturbative and independent starting point. Its origin can be traced back to the ideas of large-N reduction [4, 5], and it naturally arises from several different points of view¹.

There are very different approaches even within this matrix model, leading to considerable confusions. In this paper we focus on a weakly coupled regime on non-trivial backgrounds, to be contrasted with holographic approaches. In the present approach, the IKKT model leads to a non-standard mechanism for gravity in 3 + 1 dimensions, which circumvents the landscape of string compactification. Much progress has been made in recent years, and many of the underlying hypotheses of this approach can now be justified to some extent. The aim of this paper is to resolve some of the lingering confusions and to clarify its scope as a potential basis for fundamental physics.

From a holographic point of view following the example of AdS-CFT [6], the matrix model is viewed as a non-perturbative definition of an emergent 9 + 1 dimensional target space geometry². This is relevant in the strongly coupled (deep quantum) regime and has been substantiated for the polarized IKKT model [8–11], associating some 9+1-dimensional supergravity backgrounds to various solutions of the matrix model. The pertinent matrix model observables in that approach are of the type $\text{tr}(\mathbf{T} \dots \mathbf{T})$, viewed as geometric target space observables.

In contrast, the present approach is focused on the 3 + 1-dimensional effective geometry and physics of some **non-trivial matrix background $\mathbf{T}$ at weak coupling**³. The 9 + 1 dimensional target space is unphysical in the weakly coupled regime (similar as in $\mathcal{N} = 4$ SYM), since all perturbation modes propagate on the 3+1-dimensional background, and cannot escape into target space. Nevertheless, the background might be interpreted as a brane in target space in some decoupling limit of string theory [13]. The aim is to elaborate the physics of fluctuations on interesting such backgrounds. This approach is now sufficiently developed to warrant an introductory overview paper.

¹The model is nothing but the dimensional reduction of $\mathcal{N} = 1$ SUSY Yang-Mills to a point, and it arises also naturally from the perspective of noncommutative gauge theory.

²For the BFSS model [7], this leads to a relation with some black hole geometry in supergravity.

³The distinction between strongly coupled and weakly coupled regimes does make sense on a non-trivial background, even though there is no free parameter in the IKKT model [12].

More specifically, we consider non-trivial 3+1-dimensional matrix backgrounds or vacua $\mathbf{T}$, where the matrix fluctuations $\mathbf{T} + \mathcal{A}$ are weakly coupled and small. The reduction to 3+1 dimensions is hence achieved by vacuum selection, hence by spontaneous symmetry breaking (SSB). The landscape problem in string theory is avoided thereby, because there is no need for compactification; in fact compactification would destroy the non-perturbative power of the model. This idea has been extensively tested and is supported by non-perturbative studies of the IKKT model and similar matrix models⁴ [14–18], providing growing evidence for SSB towards 3+1 dimensions. Novel methods to tackle Yang-Mills matrix models using quantum computers are also being developed [19, 20].

It should be clear that "vacuum selection" is a dynamical process within the model, which should have a definite answer eventually. We describe an explicit quantum spacetime describing an expanding homogeneous and isotropic FLRW spacetime, which is a solution of the one-loop effective action with remarkably nice physical properties, given its simplicity.

The vacua under consideration describe quantum space-time through a noncommutative algebraic structure interpreted in terms of quantized symplectic spaces. This is familiar from quantum mechanics, where phase space is described through non-commuting matrices X and P . In the matrix model, such backgrounds lead to a noncommutative gauge theory, which has been discussed extensively in the literature from various points of view [21–24]. While this was largely abandoned in the 2000's due to UV/IR mixing, that phenomenon is now recognized as a manifestation of the stringy nature of the theory, which is benign in the maximally supersymmetric IKKT model.

Although the present paper contains some new results and insights (notably on the massless sector in section 6.2 and on time & unitarity in section 4.1), the main aim is to provide a coherent, accessible and up-to-date discussion of the framework and its basic mechanisms. Not all details are included for the sake of brevity, but links are provided for the detailed references, including [25]. The emphasis is on the physics aspects, pointing to a path towards real physics.

2 Euclidean gauge theory from matrix models

This section serves as a warm-up, recalling how noncommutative (NC) gauge theory arises from Yang-Mills-type matrix models in non-trivial matrix vacua.

Euclidean Yang-Mills matrix models are defined by an action of the form

$$S_{\text{YM}}[\mathbf{T}, \Psi] = \text{tr} \left(-\frac{1}{4g^2} [\mathbf{T}^a, \mathbf{T}^b] [\mathbf{T}_a, \mathbf{T}_b] + \frac{\mu^2}{2} \mathbf{T}^a \mathbf{T}_a + \bar{\Psi} \Gamma^a [\mathbf{T}_b, \Psi] \right) \quad (1)$$

where $\mathbf{T}_a \in \text{End}(\mathcal{H})$, $a = 1, \dots, D$ are hermitian matrices transforming as vectors under $SO(D)$, and Ψ is a matrix-valued $SO(D)$ spinor. Here $\mathcal{H} = \mathbb{C}^N$, but will also consider infinite-dimensional matrices with $\mathcal{H}$ a separable Hilbert space. Cubic term such as $\text{tr}(f_{abc} \mathbf{T}^a [\mathbf{T}^b, \mathbf{T}_c])$ may also be included. This action is invariant under a "global" $SO(D)$ symmetry acting on the indices, as well as the fundamental $U(\mathcal{H})$ gauge invariance

$$\mathbf{T}^a \rightarrow U^{-1} \mathbf{T}^a U, \quad \Psi \rightarrow U^{-1} \Psi U. \quad (2)$$

⁴Numerical studies are highly challenging due to the oscillatory nature of the matrix integral.

Gauge invariance means that gauge equivalent matrices are identified. Hence bosonic physical configurations are gauge orbits of matrices $\{\mathbf{T}^a\}_{/\sim}$ denoted as **matrix configurations**.

The quantization of Euclidean matrix models is naturally defined by integrating over the space of all matrices, which respects all symmetries. Then correlators are defined as

$$\langle \mathcal{O}(\mathbf{T}) \rangle := \frac{1}{Z} \int d\mathbf{T} d\Psi \mathcal{O}(\mathbf{T}) e^{-S[\mathbf{T}]}, \quad Z = \int d\mathbf{T} d\Psi e^{-S[\mathbf{T}]} \quad (3)$$

where $\mathcal{O}(\mathbf{T})$ is some matrix observable, i.e. some function of the matrices $\mathbf{T}^a$. This integral is well-defined and absolutely convergent for $\mu^2 > 0$, and also for $\mu = 0$ under some restrictions [26, 27].

Our basic assumption or hypothesis is that for large N , there are non-trivial matrix configurations of the model which are (sufficiently) stable at the quantum level to support interesting physics. This will be indicated by

$$\boxed{\langle \mathbf{T}^a \rangle = \bar{T}^a} \quad (4)$$

where $\bar{T}^a \in \text{End}(\mathcal{H})$ is some non-trivial matrix configuration, i.e. a set of $9+1$ hermitian matrices (modulo gauge transformations); the bar will often be omitted. Such a matrix configuration will be considered as **vacuum** or **background**.

In general, there can be several possible such vacua. The most ambitious goal is to understand whether there is a dominant such vacuum; this can be studied in non-perturbative approaches. We will take a more modest perturbative point of view, where the matrix integral is viewed – and evaluated – as an integral of fluctuations around the background (or rather its gauge orbit). The background under consideration is required to be sufficiently stable in this perturbative sense. In particular, the typical quantum fluctuations $\mathbf{T}^a \rightarrow \mathbf{T}^a + \mathcal{A}^a$ of the matrices in (3) should be much smaller than the background:

$$\Delta^{(Q)} \mathcal{A}^a := \sqrt{\langle |\mathcal{A}^a|^2 \rangle} \ll |\mathbf{T}^a| \quad \text{semi-classical regime} \quad (5)$$

for components a with non-trivial background. In that case, noncommutative backgrounds with $[T^a, T^b] \neq 0$ can be interpreted as **quantum space**, which admit a semi-classical regime. The $U(\mathcal{H})$ gauge symmetry is then realized non-linearly on the fluctuations $\mathcal{A}$, which are governed by a gauge field theory on the quantum space defined by the background. Space(time) and fields are thus treated on the same footing. These two distinct notions of "quantum" on noncommutative backgrounds should not be confused.

It is not hard to see [12] that the above criterion (5) typically holds in the weak coupling regime (as defined by the background), in contrast to the strong-coupling deep quantum regime typically interpreted in terms of holography.

Given such a quantum space, the matrices are naturally organized in harmonics $\Phi = \sum c_\Lambda \Upsilon_\Lambda$, defined as solutions of

$$\square \Upsilon_\Lambda = \lambda_\Lambda \Upsilon_\Lambda, \quad \square = [\mathbf{T}^a, [\mathbf{T}_a, \cdot]] \quad (6)$$

where $\square$ is the matrix Laplacian. These harmonics $\{\Upsilon_\Lambda\}$ forms an ON-basis of $\text{End}(\mathcal{H})$. The tangential fluctuations $\mathcal{A}^a$ can be similarly decomposed into vector-valued harmonics, in the spirit of background field theory.

Example: the ARS model

To illustrate the above statements, consider the ARS model [28] defined by the action

$$S_{\text{ARS}}[\mathbf{T}] = N \text{tr} \left(-\frac{1}{4} [\mathbf{T}^a, \mathbf{T}^b] [\mathbf{T}_a, \mathbf{T}_b] - \frac{i\alpha}{3} \varepsilon^{abc} \mathbf{T}_a \mathbf{T}_b \mathbf{T}_c + \frac{\mu^2}{2} \mathbf{T}^a \mathbf{T}_a \right) \quad (7)$$

for 3 hermitian $N \times N$ matrices $\mathbf{T}_a$, $a = 1, 2, 3$. Here α controls the cubic (Myers or Chern-Simons) term⁵. This action enjoys a "global" $SO(3)$ symmetry acting on the indices, as well as a $SU(N)$ gauge invariance. Saddle points of the action obey the classical equations of motion (eom)

$$\square \mathbf{T}_a = i\alpha \varepsilon_{abc} \mathbf{T}^b \mathbf{T}^c - \mu^2 \mathbf{T}_a, \quad \square = [\mathbf{T}^a, [\mathbf{T}_a, \cdot]] \quad (8)$$

which – for suitable parameters – admit fuzzy spheres with suitable radius r as solution:

$$\mathbf{T}_a = r L_a, \quad [L_a, L_b] = i \varepsilon_{abc} L_c, \quad (9)$$

with L_a a representation of $su(2)$. Stability of such a solution is obvious for a choice of parameters where the action takes the form

$$S_{\text{ARS}} = N \text{tr} \left(\left(T^a - \frac{i}{r} \varepsilon^{abc} T_b T_c \right) \left(T_a - \frac{i}{r} \varepsilon_{ade} T^d T^e \right) \right) \quad (10)$$

and S is positive definite (this extends more generally to some range in parameter space). The global minimum $S = 0$ is then assumed for any background of the form

$$\mathbf{T}_a = \oplus_i r L_a^{(N_i)} \quad (11)$$

corresponding to a stack of fuzzy spheres $S_{N_i}^2$ with N_i with $\sum_i N_i = N$. These are candidates for the vacua (4). The $SU(N)$ gauge symmetry still applies in such a vacuum (which is really an entire $SU(N)$ gauge orbit), but it is realized non-linearly on the fluctuations, leading to a noncommutative Yang-Mills gauge theory. The global $SO(3)$ symmetry is also unbroken because it is equivalent to a gauge transformation,

$$\Lambda_b^a \mathbf{T}^b = U^{-1} \mathbf{T}^a U \quad (12)$$

for $\Lambda \in SO(3)$ with representation $U(\Lambda) \in SU(N)$. Such backgrounds are denoted as **covariant quantum spaces**.

Low-energy fluctuations on a stack of k coincident fuzzy spheres S_n^2 with $N = kn$ organize into tangential Yang-Mills $U(k)$ gauge fields on S_n^2 , similar as on $\mathbb{R}_\theta^2$ discussed in the next section, while transversal fluctuations correspond to a scalar field, which plays the role of a Higgs field if the N_i are different. In addition there are non-local high energy "stringy" fluctuation modes, which have no classical analog [29].

At the quantum level i.e. under the matrix integral, one typically finds a nontrivial "fuzzy sphere" phase with a vacuum as above, and a "matrix" phase of random matrix fluctuations around the trivial vacuum, depending on the parameters [30–33]. In the fuzzy-sphere phase, the fluctuation spectrum matches the expected decomposition into fuzzy spherical harmonics. Instabilities may arise for low harmonics, triggering collapse to the commuting phase. An interesting question is whether there is one dominant vacuum at the quantum level, or many possible vacua at the same footing. Monte Carlo simulations indicate coalescence into larger irreps at sufficiently large α , consistent with analytic one-loop calculations [33].

⁵The normalization of the Yang-Mills term is immaterial, since it can be absorbed in the matrices.

2.1 Semi-classical regime and geometry

We return to the general discussion of matrix configurations. Any given $\mathbf{T}^a$ defines a set of derivations

$$e^a = -i[\mathbf{T}^a, \cdot] \quad (13)$$

on the space of matrices $\text{End}(\mathcal{H})$, which respect the (Hilbert-Schmidt) inner product $\text{tr}(A^\dagger B)$. Hence these vector fields always define unitary 1-parameter evolution operators

$$e^{t[\mathbf{T}^a, \cdot]} \quad (14)$$

acting on the space of matrices $\text{End}(\mathcal{H})$.

For almost-commuting matrix configurations i.e. $[\mathbf{T}^a, \mathbf{T}^b] \ll \mathbf{T}^a \mathbf{T}^b$ in some matrix norm⁶, the algebra of matrices can typically be related to the algebra of functions on some underlying manifold $\mathcal{M}$,

$$\begin{aligned} \text{End}(\mathcal{H}) &\sim \mathcal{C}(\mathcal{M}) \\ \Phi &\sim \phi \\ [\cdot, \cdot] &\sim i\{\cdot, \cdot\} \\ \text{Tr}(\Phi) &\sim \int \Omega \phi. \end{aligned} \quad (15)$$

Then the commutator reduces to a Poisson bracket, which define a symplectic structure in the non-degenerate case, with symplectic volume form Ω . This *semi-classical* or Poisson description is typically accurate for long wavelengths, above some scale L_{NC} of noncommutativity. The derivations (13) then define a set of canonical vector fields on $\mathcal{M}$,

$$\begin{aligned} e^a &= -i[\mathbf{T}^a, \cdot] \sim \{\mathbf{T}^a, \cdot\} \\ e^{a\mu} &\sim \{\mathbf{T}^a, x^\mu\} \end{aligned} \quad (16)$$

which play the role of a **frame**⁷; here x^μ are local coordinate functions on $\mathcal{M}$.

On the other hand, due to the relation $\text{End}(\mathcal{H}) \sim \mathcal{C}(\mathcal{M})$ the matrices $\mathbf{T}^a$ can also be interpreted as quantized embedding functions

$$\mathbf{T}^a \sim \mathbf{t}^a : \mathcal{M} \hookrightarrow \mathbb{R}^D \quad (17)$$

of $\mathcal{M}$ into target space. These two interpretations of the matrices are perfectly consistent: observables in classical or quantum mechanics can be viewed either as functions or as generators of canonical transformations. Both points of view provide complementary insights.

As usual in gravity, the frame determines (up to a conformal factor) an effective metric, which governs the kinetic term of fluctuation modes on the background [25, 36, 37]:

$$G_{\mu\nu} := \rho^2 \gamma_{\mu\nu}, \quad \gamma^{\mu\nu} = \eta_{ab} e^{a\mu} e^{b\nu} \quad (18)$$

$$\rho^2 = \rho_M \sqrt{|\gamma^{\mu\nu}|} = \rho_M \det(e^{\dot{a}\mu}) = \frac{\sqrt{|G|}}{\rho_M}. \quad (19)$$

⁶This can be the Hilbert-Schmidt norm, or a more local measure in terms of expectation values of quasi-coherent states; see [12, 34] for a more detailed discussion.

⁷An early precursor was proposed in [35], however with algebraic constraints. The frame arises automatically from the matrix-model framework, without these constraints.

Here $\Omega = d^4x \rho_M$ is the symplectic volume form on $\mathcal{M}$, which we assumed to be 4-dimensional to be specific. This it is easily derived from the quadratic action governing the transversal matrices $T^i \equiv \phi$, which play the role of scalar fields. That metric turns out to govern the kinetic action for *all* fluctuations on the matrix background, therefore it must be interpreted in terms of gravity. The metric is also encoded in the matrix Laplacian

$$\square = [\mathbf{T}^a, [\mathbf{T}_a, \cdot]] \sim \rho^2 \square_G \quad (20)$$

where $\square_G$ is the Laplace-Beltrami operator for $G_{\mu\nu}$. The matrix harmonics (6) thus becomes eigenfunctions of $\square_G$ on $\mathcal{M}$. For example, the harmonics of $\square$ for the background (9) are known as fuzzy spherical harmonics.

In Euclidean signature, the effective metric can also be extracted from a Connes-type distance formula [38] for $\not{D} = \Gamma^a [\mathbf{T}_a, \cdot]$.

A matrix configuration provides further structure, notably a notion of optimally localized quasi-coherent states $|x\rangle$ [34], defined as ground states of the displacement Hamiltonian

$$H_x = \sum_a (X_a - x_a \mathbb{1})^2, \quad H_x |x\rangle = \lambda(x) |x\rangle \quad (21)$$

for $x \in \mathbb{R}^D$. These states allow to extract local properties such as fields $\phi(x) = \langle x | \Phi | x \rangle$, and to reconstruct $\mathcal{M}$ with its Berry connection $iA = \langle x | d | x \rangle$ and symplectic structure $\omega = dA$ from the set of quasi-coherent states for a given matrix background [34]. This provides useful information about the local structure of a quantum space, and also topological invariants such as Chern numbers

$$c_1 := \int_{S^2} \frac{\omega}{2\pi} \quad (22)$$

which are applicable in more general contexts [39].

2.2 Noncommutative Yang-Mills in the example of $\mathbb{R}_\theta^4$

As often in physics, taking the limit $N \rightarrow \infty$ leads to a simplification. Then the model (1) without mass term admits the following background

$$\mathbf{T}^\alpha = X^\alpha, \quad [X^\alpha, X^\beta] = i\theta^{\alpha\beta} \mathbb{1} \quad (23)$$

for $\alpha = 1, \dots, 4$; the remaining matrices $\mathbf{T}^i$ for $i = 5, \dots, D$ vanish. The classical eom $\square \mathbf{T}^a = 0$ is clearly satisfied. This background admits a description (15) as Moyal-Weyl quantum plane $\mathbb{R}_\theta^4$ with a constant Poisson tensor $\theta^{\alpha\beta}$, via the (Weyl) quantization map

$$\begin{aligned} \mathcal{Q}: \quad \mathcal{C}(\mathbb{R}^4) &\rightarrow \text{End}(\mathcal{H}) \\ e^{ikx} &\mapsto e^{ikX} \end{aligned} \quad (24)$$

Functions in X^μ are thereby interpreted as scalar fields on $\mathbb{R}_\theta^4$. Now consider fluctuations $\mathbf{T}^a = \bar{T}^a + \mathcal{A}^a$ on this vacuum

$$\bar{T}^a = \begin{pmatrix} X^\alpha \\ 0 \end{pmatrix}, \quad \alpha = 1, \dots, 4. \quad (25)$$

To understand the dynamics of the fluctuations, we rewrite them as

$$\mathcal{A}^{\dot{a}} = \begin{pmatrix} \mathcal{A}^\alpha \\ \mathcal{A}^i \end{pmatrix} = \begin{pmatrix} \theta^{\alpha\mu} A_\mu \\ \phi^i \end{pmatrix}. \quad (26)$$

One then finds using $[X^\alpha, \cdot] = i\theta^{\alpha\mu}\partial_\mu$

$$\begin{aligned} [\mathbf{T}^\alpha, \phi] &= [X^\alpha + \mathcal{A}^\alpha, \phi] = i\theta^{\alpha\mu}(\partial_\mu\phi - i[A_\mu, \phi]) =: i\theta^{\alpha\mu}D_\mu\phi, \\ [\mathbf{T}^\alpha, \mathbf{T}^\beta] &=: -i\mathcal{F}^{\alpha\beta} = i\theta^{\alpha\mu}\theta^{\beta\nu}(-\theta_{\mu\nu}^{-1} + F_{\mu\nu}) \end{aligned} \quad (27)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ is the $U(1)$ field strength on $\mathbb{R}_\theta^4$. Then the bosonic action takes the semi-classical form of a noncommutative Yang-Mills gauge theory on $\mathbb{R}_\theta^4$

$$S[A, \phi] = \int_{\mathbb{R}_\theta^4} \frac{d^4x}{(2\pi)^2} \sqrt{|G|} \left(-\frac{1}{g_{\text{YM}}^2} F_{\mu\nu} F^{\mu\nu} - 2D^\mu\phi^i D_\mu\phi_i + g_{\text{YM}}^2 [\phi^i, \phi^j][\phi_i, \phi_j] \right) \quad (28)$$

where covariant indices are contracted with $G_{\mu\nu}$, and the effective coupling constant

$$g_{\text{YM}}^2 = \rho^{-2} \quad (29)$$

is assumed to be small. The gauge symmetry acts non-linearly on the fluctuations as

$$A_\mu \rightarrow U^{-1} A_\mu U + iU^{-1} \partial_\mu U, \quad \phi^i \rightarrow U^{-1} \phi^i U. \quad (30)$$

This extends straightforwardly to a $U(k)$ Yang-Mills theory upon replacing the background by a stack of coinciding $\mathbb{R}_\theta^4$ "branes"

$$\bar{T}^a = \begin{pmatrix} X^\alpha \otimes \mathbb{1}_k \\ 0 \end{pmatrix}. \quad (31)$$

Despite appearance, the $U(1)$ sector of this theory – which would be free on ordinary space-time – is now geometrical, since it changes the background and the effective metric $G_{\mu\nu}$.

3 The Lorentzian IKKT model

In order to obtain spacetime, we must embrace Lorentzian matrix models, and notably the Lorentzian IKKT model which is defined by

$$S_{\text{IKKT}} = \text{Tr} \left(\frac{1}{4} [\mathbf{T}_a, \mathbf{T}_b] [\mathbf{T}^a, \mathbf{T}^b] + \bar{\Psi} \Gamma^a [\mathbf{T}_a, \Psi] \right) \quad (32)$$

where $\mathbf{T}_a$, $a = 0, \dots, 9$ are Hermitian matrices, and Ψ are matrix-valued Majorana-Weyl spinors of $SO(9, 1)$. The quantization – defined by integrating over the space of all matrices – now amounts to an oscillatory integral, which requires regularization through a suitable $i\varepsilon$ prescription indicated by S_ε . Then correlators are then defined as

$$\langle \mathcal{O}(T) \rangle := \frac{1}{Z_\varepsilon} \int dT d\Psi \mathcal{O}(T) e^{iS_\varepsilon[T]}, \quad Z_\varepsilon = \int dT d\Psi e^{iS_\varepsilon[T]} \quad (33)$$

One way to approach this is via a Wick rotation such as $\mathbf{T}^0 \rightarrow e^{-i3\pi/8}\mathbf{T}^0$, $\mathbf{T}^i \rightarrow e^{i\pi/8}\mathbf{T}^i$ (along with a suitable phase factor for the action, cf. [14]). This is useful notably for the trivial vacuum, and it allows to relate Euclidean observables⁸ such as $\text{Tr}(\mathbf{T}_a\mathbf{T}^b)$ to Wick-rotated Lorentzian observables via contour integration. However, a Wick rotation does not preserve hermiticity of the matrices, and it misses the Lorentzian saddle points of interest⁹. Therefore a full Wick rotation is not useful for the present approach.

A better way to define the Lorentzian model is via an "infinitesimal Wick rotation"

$$\mathbf{T}^0 \rightarrow e^{-i\varepsilon}\mathbf{T}^0, \quad \mathbf{T}^i \rightarrow e^{i\varepsilon}\mathbf{T}^i \quad (34)$$

or similar. This leads to Feynman's $i\varepsilon$ prescription for the emergent field theory on Lorentzian saddle points, and it contributes exponential damping to the matrix integral. However, the action still has flat directions for commuting matrices. We remove those by adding an extra regularizing mass term

$$S_\varepsilon = S + \text{tr}\left(i\varepsilon \sum_a (\mathbf{T}^a)^2\right) \quad (35)$$

slightly breaking the global $SO(9,1)$. This leads to an explicit Gaussian damping of the Lorentzian matrix integral which is now absolutely convergent, even in the presence of the Pfaffian obtained by integrating out the fermionic matrices. The Lorentzian model is then no longer related to the Euclidean model by Wick rotation. Including a (small) mass term may also help to select a particular (type of) saddle point¹⁰, cf. [14].

3.1 Vacua, frame and geometry

We follow again a perturbative approach, focusing on some non-trivial matrix configuration $\langle \mathbf{T}^a \rangle = \bar{T}^a$ (4) of the model which is sufficiently stable for large $N = \dim \mathcal{H}$. Such vacua are known to exist in the $N \rightarrow \infty$ limit, where the IKKT model admits many different saddle points such as $\mathbb{R}_\theta^{2n}$. It is thus plausible that for large but finite N , the model admits – either at the quantum level or upon adding a small mass term – a number of distinct saddle points or vacua. We want to understand the physics of fluctuations around such a vacuum.

Non-trivial vacua should be saddle points $\{\mathbf{T}^a\}$ of the quantum effective action. Candidates for such backgrounds are given by solutions of the classical matrix equations of motion

$$\square \mathbf{T}^a = 0, \quad \square = [\mathbf{T}^a, [\mathbf{T}_a, \cdot]] \quad (36)$$

For finite-dimensional matrices, these equations have no solutions¹¹, but solutions do exist¹² for the mass-deformed equations of motion with positive mass,

$$\square \mathbf{T}^a = \mu^2 \mathbf{T}^a, \quad \mu^2 > 0 \quad (37)$$

⁸Such target space observables are the focus of the holographic approach, aiming to extract an effective target space metric at strong coupling.

⁹Euclidean saddle points would correspond to instanton-type configurations in the Lorentzian models.

¹⁰Another interesting approach is to "gauge-fix" the non-compact $SO(9,1)$ symmetry [40]

¹¹This follows from the fact that $\square$ is positive semi-definite if acting on T^0 , cf. [41].

¹²for example, $\mathbf{T}^0 = \sqrt{2}\lambda_4$, $\mathbf{T}^1 = \lambda_1$, $\mathbf{T}^2 = \lambda_2$ for $\mathfrak{su}(3)$ generators λ_a gives a solution with $m^2 = 2$.

In the undeformed IKKT model with $\mu = 0$, such Lorentzian saddle points are expected to be stabilized by quantum effects [42].

For $N = \infty$, one background of the IKKT model is given by $\mathbb{R}_\theta^4(31)$, where the model leads to maximally supersymmetric noncommutative $\mathcal{N} = 4$ SYM as in section 2.2; that background is hence BPS protected. Analogous statements hold for $\mathbb{R}_\theta^{2n}$ for any $2n \leq 10$. This leads to an important corollary: at the non-perturbative level, noncommutative $\mathcal{N} = 4$ SYM on $\mathbb{R}_\theta^4$ is **the same theory** as the maximally SUSY gauge theory on any $\mathbb{R}_\theta^{2n}$ which would arise from the IKKT model on that background, for $N = \infty$. This fact is not visible in the perturbative weak coupling regime of the gauge theory¹³, analogous to the different vacua encountered in the ARS model.

A related observation is that fluctuations around a vacuum describing some quantum space $\mathcal{M}$ (e.g. with $\dim \mathcal{M} = 3 + 1$) are automatically confined to $\mathcal{M}$ at weak coupling. Indeed all matrix fluctuations play the role of (local or non-local) functions on $\mathcal{M}$ due to (15), and there are simply no modes moving in the $9 + 1$ -dimensional target space outside of $\mathcal{M}$. The matrix fluctuations around the background are governed by a Yang-Mills type action, and the matrix integral (32) amounts to a path integral quantization of that field theory, with weight $e^{\frac{i}{\hbar}S}$. Since the IKKT model is related to IIB string theory, this mechanism provides an alternative to string compactifications and their lack of predictivity.

A priori, it is not clear whether one saddle point will dominate the matrix integral, or many different saddle points (possibly with different dimensions) contribute significantly, with complex weights. Although this is not essential for the physics on some (interesting) background, it would be desirable to show that the "dominant" vacua are $3+1$ dimensional. There is growing evidence from numerical simulations that this is indeed the case [14–18]; for attempts to understand this analytically see e.g. [43–46]. Within the present perturbative approach, we can offer the following argument: An almost-local quantum effective action can arise only for backgrounds with dimension ≤ 4 , since the YM theory would be UV divergent in higher dimensions, signaling an instability of the background. This would lead to pathological non-locality in the effective action due to UV/IR mixing via non-local string modes $|x\rangle\langle y|$ running in the loops. On the other hand, spacetimes with dimension < 4 are expected to be insignificant under the matrix integral. However, settling this question will presumably require new techniques, cf. [47–49] for possibly relevant developments.

3.2 Geometric structure: Weitzenböck connection and torsion

From now on we focus on $3+1$ dimensional backgrounds $\mathbf{T}^{\dot{\alpha}}$. To avoid confusion with coordinate indices, we will often denote the $3 + 1$ matrix or frame indices with dotted greek letters, indicating that they transform as vectors under the global $SO(3, 1)$ symmetry.

As in the Euclidean case, the background defines a frame $e^{\dot{\alpha}} = \{\mathbf{T}^{\dot{\alpha}}, .\}$ (16) in the semiclassical regime, and thereby an effective metric. This frame is divergence-free due to the Jacobi identity; in particular, there is no frame bundle. Instead of working with the Levi-Civita connection, it is then more natural to work with the unique connection which respects the given frame, known as Weitzenböck connection:

$$\nabla^{(W)} e^{\dot{\alpha}} = 0; \quad (38)$$

¹³for $n = 0$, these are point-like extreme UV solitons in $\mathcal{N} = 4$ NC SYM known as fluxons [24].

we assume here that the matrix background defining spacetime is $3 + 1$ dimensional. This connection is metric-compatible and flat, and its torsion encodes the same information as the standard spin connection. The torsion can be expressed via the bracket structure as follows [50]

$$\begin{aligned}\mathcal{T}[e_{\dot{\alpha}}, e_{\dot{\beta}}] &= \nabla_{\dot{\alpha}}^{(W)} e_{\dot{\beta}} - \nabla_{\dot{\beta}}^{(W)} e_{\dot{\alpha}} - [e_{\dot{\alpha}}, e_{\dot{\beta}}] = -[e_{\dot{\alpha}}, e_{\dot{\beta}}] \\ &\equiv -\{\mathbf{T}_{\dot{\alpha}}, \{\mathbf{T}_{\dot{\beta}}, \cdot\}\} + \{\mathbf{T}_{\dot{\beta}}, \{\mathbf{T}_{\dot{\alpha}}, \cdot\}\} \\ &= \{\mathcal{F}_{\dot{\alpha}\dot{\beta}}, \cdot\}, \quad \mathcal{F}_{\dot{\alpha}\dot{\beta}} = i[\mathbf{T}_{\dot{\alpha}}, \mathbf{T}_{\dot{\beta}}] \sim -\{\mathbf{T}_{\dot{\alpha}}, \mathbf{T}_{\dot{\beta}}\}\end{aligned}\tag{39}$$

using the Jacobi identity, hence

$$\mathcal{T}^{\dot{\alpha}\dot{\beta}\mu} = \{\mathcal{F}^{\dot{\alpha}\dot{\beta}}, x^{\mu}\} .\tag{40}$$

That formula underlies the importance of the Weitzenböck connection in matrix models, and it is crucial to obtain the induced Einstein-Hilbert action (83).

For covariant quantum spacetime discussed below, the dimension of $\mathcal{M}$ will be larger than the number of background matrices, leading to a higher-spin structure of the torsion.

3.3 Vacuum energy: evading the cosmological constant problem

In general relativity, the Einstein equations

$$\mathcal{G}_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}\tag{41}$$

describes the gravitational dynamics of massive objects. However they also lead to the notorious cosmological constant problem, as vacuum energy curves spacetime. Vacuum energy is generically large on non-supersymmetric backgrounds, since it includes the zero-point energy due to quantum fluctuations of all modes below the SUSY breaking scale. This entails a large curvature of spacetime, in conflict with observation. The only way to avoid this clash within GR is to postulate some finely tuned a-priori cosmological constant Λ . The associated unreasonable fine-tuning constitutes the cosmological constant problem.

In the present framework, this problem does not arise, because the Einstein equations are only valid on intermediate scales. At very large scales, the geometry of spacetime is determined by the combination of the Yang-Mills action with vacuum energy, while the Einstein-Hilbert action is subleading and becomes irrelevant. Then asymptotically flat spacetimes can be perfectly consistent with large vacuum energy, as illustrated below.

3.4 Covariant quantum spacetime

Covariant quantum spaces or spacetimes are matrix backgrounds which enjoy covariance in the sense of (12), under a sufficiently large and interesting covariance group. Several interesting candidates have been discussed in the literature, including fuzzy S_N^4 , the fuzzy hyperboloid H_n^4 , and covariant quantum spacetime $\mathcal{M}_n^{3,1}$. We will focus on the latter because it provides a near-realistic candidate for physical spacetime, albeit with $\dim \mathcal{H} = \infty$.

3.4.1 Rigid cosmological background

Now we discuss an interesting prototype for a matrix background denoted as covariant quantum spacetime. Let M^{AB} , $A, B = 0, \dots, 5$ be the hermitian generators of $\mathfrak{so}(4, 2) \cong \mathfrak{su}(2, 2)$ in the minimal unitary "doubleton" representation¹⁴ $\mathcal{H}_0$, and define

$$T^\mu = r^{-1} M^{\mu 4}, \quad X^a = r M^{a5} \quad (42)$$

for $\mu = 0, \dots, 3$ and $a = 0, \dots, 4$. Due to the special choice of representation, these generators satisfy the constraints

$$X_\mu X^\mu = r^2 \mathbb{1} - X_4^2, \quad (43a)$$

$$T_\mu T^\mu = -\frac{1}{r^2} \mathbb{1} + \frac{X_4^2}{r^4}, \quad (43b)$$

$$X_\mu T^\mu + T_\mu X^\mu = 0. \quad (43c)$$

The space of matrices $\mathcal{C} := \text{End}(\mathcal{H}_0)$ can be identified as (functions on) quantized twistor space $\mathcal{M} \cong \mathbb{CP}^{1,2}$, which is a coadjoint orbit of $SU(2, 2)$. It decomposes into subsectors

$$\text{End}(\mathcal{H}_0) =: \mathcal{C} = \bigoplus_{s=0}^{\infty} \mathcal{C}^s \sim \mathcal{C}(\mathcal{M}). \quad (44)$$

Here $\mathcal{C}^s$ denotes the eigenspace of an $SO(4, 1)$ Casimir $\mathcal{S}^2$ with eigenvalue $2s(s+1)$ [37]; in particular, $\mathcal{C}^0$ can be identified with completely symmetrized functions of X^μ [51]. This quantum space has **finitely many degrees of freedom per volume**, which follows from the discrete structure of the lowest-weight Hilbert space $\mathcal{H}_0$ shown in figure 1. For example,

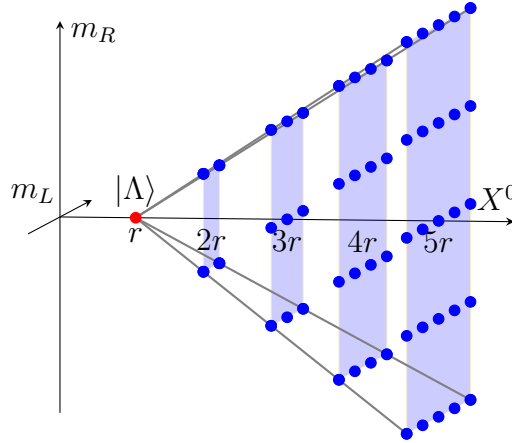


Figure 1: Weight structure of $\mathcal{H}_0$, with $m_{L,R}$ denoting $SU(2)_L$ and $SU(2)_R$ weights.

the number of states in $\mathcal{H}_{0,N} := \{|\psi\rangle \in \mathcal{H}_0; X^0 \leq NR\}$ with $X^0 \leq NR$ is given by the

¹⁴This representation can be explicitly constructed in terms of 4 harmonic oscillator algebras. There is also a series of generalized doubleton irreps $\mathcal{H}_n$ with $n \in \mathbb{N}$ with very similar properties, cf. [51, 52].

symplectic volume of the corresponding subspace of $\mathcal{M}$ [53]

$$\dim \mathcal{H}_{0,N} \sim \frac{1}{3} N^3 \sim \int_{x_0 \leq NR} \Omega \quad (45)$$

for large N . This implies also an effective upper limit for localized $\mathfrak{hs}$ modes

$$s \leq s_{max} = r^{-1} X^0 . \quad (46)$$

The commutation relations¹⁵ for the minimal $n = 0$ case can be written in closed form

$$[T^\mu, X^\nu] = \frac{i}{r} X_4 \eta^{\mu\nu} \quad (47a)$$

$$[X^\mu, X^\nu] = -ir X_4^{-1} (T^\mu X^\nu - T^\nu X^\mu) , \quad (47b)$$

$$[T^\mu, T^\nu] = i X_4^{-1} (T^\mu X^\nu - T^\nu X^\mu) \quad (47c)$$

where $X_4^2 = r^2 - X_\mu X^\mu$. The T^μ are eigenvectors of $\square_T = [T^\mu, [T_\mu, .]]$ with positive eigenvalue

$$\square T^\mu = \frac{3}{r^2} T^\mu , \quad (48)$$

which can be viewed as equations of motion of the matrix model with mass term. Note that a positive eigenvalue of $\square$ indicates that this background should admit finite-dimensional deformations with similar structure, according to the discussion around (37). This is not the case for the background X^μ , which satisfies $\square_X X^\mu = -3r^2 X^\mu$.

We thus consider the following background for the IKKT model

$$\mathbf{T}^{\dot{\alpha}} = T^{\dot{\alpha}}, \quad \dot{\alpha} = 0, \dots, 3 \quad (49)$$

which is very interesting for several reasons. Most importantly, $SO(3, 1)$ covariance

$$\Lambda^{\dot{\alpha}}_{\dot{\beta}} \mathbf{T}^{\dot{\beta}} = U^{-1} \mathbf{T}^{\dot{\alpha}} U \quad (50)$$

means that $SO(3, 1)$ is fact unbroken in the matrix model. This will imply exact homogeneity and isotropy, even though there are only finitely many degrees of freedom per volume. That background will be slightly generalized below to take into account the dynamics, and the uniqueness statement in section 3.4.2 will guarantee existence of such a solution at the quantum level. The construction of quasi-coherent states (21) on $\mathcal{M}$ is somewhat subtle, and we refer to [53] for details.

Bundle structure over spacetime. In the semi-classical regime, the $X^\mu \sim x^\mu$ can be viewed as bundle map

$$x^\mu : \mathcal{M} \rightarrow \mathcal{M}^{3,1} \subset \mathbb{R}^{3,1} \quad (51)$$

¹⁵Although the X^μ and T^ν are $\mathfrak{so}(4, 2)$ Lie algebra generators, the closed form of the rhs depends on the doubleton representations $\mathcal{H}_n$ under consideration, and is simplest for $n = 0$.

compatible with $SO(3, 1)$. The fiber of this map is resolved by the $t^\mu \sim T^\mu$ generators, which describe a space-like S^2 due to the constraints

$$r^2 t_\mu t^\mu = -r^{-2} x_\mu x^\mu, \quad t_\mu x^\mu = 0 \quad (52)$$

evaluated near some point $\xi^\mu = (\xi^0, 0, 0, 0)$ on the base manifold $\mathcal{M}^{3,1}$. This sphere transforms under the local $SO(3)$ stabilizer group of any point of $\mathcal{M}^{3,1}$, which plays the role of spacetime. Therefore harmonics on S^2 play the role of higher-spin ($\mathfrak{hs}$) modes on spacetime. The decomposition (44) of the algebra of functions $\mathcal{C}$ into $\mathfrak{hs}$ modes can be made explicit

$$\begin{aligned} \mathcal{C} = \mathbb{C}[[x^\mu, t_\nu]]/\sim &= \mathcal{C}^0 \oplus \mathcal{C}^1 \oplus \mathcal{C}^2 \oplus \dots \\ \mathcal{C}^s \ni \phi^{(s)} &= \phi_\mu(x) u^\mu \end{aligned} \quad (53)$$

in terms of irreducible polynomials in $u^\mu = \frac{r^2}{x_4} t^\mu$ of degree s , which generate an internal unit sphere S^2 . Due to the constraint $t_\mu x^\mu = 0$ the u^μ are space-like, so that the tensor fields $\phi_\mu(x)$ are space-like. This is crucial for the unitarity of the emergent field theory. Together with the compactness of the fiber S^2 (which is essential for UV finiteness), this is the reason why the present covariant quantum spacetimes provide good backgrounds for the matrix model, in contrast to other backgrounds proposed in the literature [54–56].

The commutation relations (47) reduce in the semi-classical regime to Poisson brackets¹⁶,

$$\begin{aligned} \{t^\mu, x^\nu\} &= \frac{x_4}{r} \eta^{\mu\nu} \\ \{x^\mu, x^\nu\} &= -\frac{r^3}{x_4} (t^\mu x^\nu - t^\nu x^\mu) =: \theta^{\mu\nu} = -r^4 \{t^\mu, t^\nu\} \end{aligned} \quad (54)$$

which allow to obtain the frame and metric as

$$e^{\dot{a}\mu} = \frac{x^4}{r} \eta^{a\mu}, \quad G_{\mu\nu} = \sinh(\tau) \eta_{\mu\nu}, \quad \rho^2 = \rho_M \det(e^{\dot{a}\mu}) = \frac{2}{r^4} \sinh^3(\tau) \quad (55)$$

in Cartesian coordinates x^μ . Here τ is a convenient time parameter τ on $\mathcal{M}^{3,1}$ defined by

$$x_4 = r \sinh(\tau) \quad (56)$$

and $\Omega = \rho_M d^4 x$ is the symplectic volume form reduced to $\mathcal{M}^{3,1}$. Note that this frame is classical i.e. without $\mathfrak{hs}$ components, and invariant under local $SO(3)$ rotations. The geometry can be recognized as a (double cover of) a $k = -1$ FLRW spacetime with Big Bounce at $x_4 = 0$ [37], and scale function

$$a(t) \sim r e^{\frac{3}{2}\tau} \sim t, \quad t \rightarrow \infty \quad (57)$$

A schematic picture of the spacetime and its bundle structure is shown in figure 2.

¹⁶There are extra terms in $\theta^{\mu\nu}$ for $n \neq 0$ corresponding to non-trivial Chern numbers (22) for the internal S^2 , which characterize the different doubleton representations $\mathcal{H}_n$.

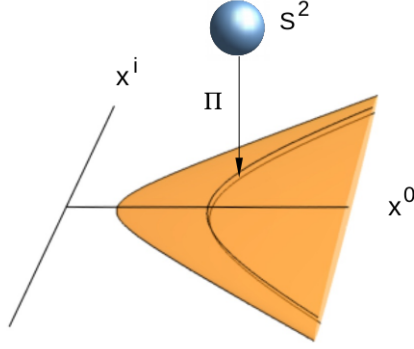


Figure 2: Sketch of the bundle space $\mathcal{M}^{3,1} \times S^2$. The black lines indicate space-like H^3 .

Rigid $k = 0$ spacetime. A very similar background leads to a FLRW cosmology with $k = 0$. It is given by the same doubleton representations of $SO(4, 2)$, but with modified matrix configuration T^μ and spacetime generators Y^μ satisfying the $E(3)$ -covariant relations

$$\begin{aligned} [T^i, T^j] &= 0, \quad [T^0, T^i] = -\frac{i}{r} T^i, \quad [T^\mu, Y^\nu] = i \frac{Y^0}{r} \eta^{\mu\nu} \\ [Y^0, Y^i] &= i r^3 T^i, \quad [Y^i, Y^j] = \frac{-i r^3}{Y^0} (T^i Y^j - T^j Y^i) \end{aligned} \quad (58)$$

(for $n = 0$) as well as the constraints

$$T^i T_i = \frac{1}{r^4} Y^{02}, \quad T^\mu Y_\mu + Y^\mu T_\mu = 0. \quad (59)$$

More details can be found in [57], and an explicit realization on $\mathcal{H} = L^2(\mathbb{R}^3)$ is given in [54]. To be specific, we will mostly focus on the $k = -1$ case in the following.

Emergent gauge theory. On covariant quantum spacetime, the action for tangential and transversal fluctuations

$$\mathcal{A}^a = \begin{pmatrix} \mathcal{A}^{\dot{a}} \\ \mathcal{A}^i \end{pmatrix} = \begin{pmatrix} e^{\dot{a}\mu} A_\mu \\ \phi^i \end{pmatrix}. \quad (60)$$

(which are now $\mathfrak{hs}$ -valued!) takes again the form of a $\mathcal{N} = 4$ SYM gauge theory

$$\begin{aligned} S_{\text{IKKT}}[A, \phi] &= \int_{\mathcal{M}^{3,1}} \frac{d^4 x}{(2\pi)^2} \sqrt{|G|} \left[-\frac{1}{g_{\text{YM}}^2} G^{\mu\mu'} G^{\nu\nu'} F_{\mu\nu} F_{\mu'\nu'} - \frac{1}{g_{\text{YM}}^2} G_{\mu\mu'} G_{\nu\nu'} B^{\mu\nu} B^{\mu'\nu'} \right. \\ &\quad \left. - 2G^{\mu\nu} D_\mu \phi^i D_\nu \phi_i + g_{\text{YM}}^2 [\phi^i, \phi^j] [\phi_i, \phi_j] \right]_0 \end{aligned} \quad (61)$$

with gauge transformations (30). Here $B^{\mu\nu} = -\frac{1}{r^2 x_4^2} \theta^{\mu\nu}$ can be interpreted as background flux. The effective coupling is again determined by the dilaton as

$$g_{\text{YM}}^2 = \frac{1}{\rho^2} \quad (62)$$

which becomes weak at late times¹⁷. This form of the action however hides the geometric nature of the theory, since $\mathcal{C}^1$ -valued gauge fields change the background and the effective metric $G_{\mu\nu}$.

A caveat should be noted here: We dropped the linear fluctuations, pretending that the background is a solution. This is true only upon adding a mass term $\mu^2 \text{tr}(\mathbf{T}^a \mathbf{T}_a)$ to the action, which is fine classically but not allowed upon quantization since it spoils SUSY. The proper way to proceed is to generalize the matrix background:

3.4.2 Dynamical covariant cosmological spacetime

The above rigid covariant spacetime is a classical solution of the model only in the presence of a mass term. However, an obvious generalization of this background gives the most general $SO(3, 1)$ -covariant matrix background (50), which will provide a solution of the proper IKKT model. Since X_4 is invariant under $SO(3, 1)$, such backgrounds are given by

$$\mathbf{T}^{\dot{\alpha}} = \alpha(X_4)T^{\dot{\alpha}} + \beta(X_4)X^{\dot{\alpha}} \quad (+h.c.), \quad (63)$$

which turns out to be the most general covariant background [59]. This uniqueness statement implies that the ansatz (63) *is guaranteed to provide a solution at the quantum level*. Moreover, one can eliminate either $\alpha(X_4)$ or $\beta(X_4)$ using a gauge transformation, and we will choose $\beta = 0$. We therefore consider the following backgrounds

$$\mathbf{T}^{\dot{\alpha}} = \alpha(X_4)T^{\dot{\alpha}} \quad (64)$$

This ansatz allows to find solutions of the classical IKKT model without mass term, however they turn out to be problematic [58]. Most importantly, physically reasonable solutions with

$$\alpha \sim x_4^{-\frac{3}{4}} \sim e^{-3\tau/4} \quad (65)$$

at late times are obtained by taking into account the 1-loop vacuum energy (85) [?]. That scaling still leads to an expanding universe with $a(t) \propto t$ at late times similar to the rigid case, but with constant dilaton, Yang-Mills coupling and UV/IR hierarchy. This is plausible because the dilaton measures the number of degrees of freedom per Riemannian volume (19); however a detailed discussion is beyond the scope of this paper. Analogous statements apply to the $k = 0$ background [57], but its evolution has not yet been worked out.

To understand the fluctuations and the physics on these backgrounds, it is necessary to find appropriate local normal coordinates for spacetime, as discussed in section 4.2.1.

4 Time evolution, Hilbert space, and unitarity

One frequently asked question is how time evolution and a physical Hilbert space of states can arise in the IKKT model, which – unlike its 0 + 1 dimensional sibling known as BFSS model [7, 60] – has no intrinsic time. We can address this question on non-trivial backgrounds¹⁸.

¹⁷This behavior can change for more general time-dependent backgrounds with the same structure [58].

¹⁸This shows the fundamental difference to holographic approaches, where everything emerges from quantum effects, often in the trivial vacuum.

Then the matrix fluctuations correspond to (off-shell, a priori) fields on spacetime governed by the matrix d'Alembertian $\square = [T^{\dot{\alpha}}, [T_{\dot{\alpha}}, \cdot]] \sim \rho^2 \square_G$ with a Lorentzian metric G , and the quantization (33) of the matrix model corresponds to the path integral quantization of QFT. Hence the Hilbert space and the causal structure of spacetime can be identified from the space of on-shell modes, and their details depend on the type of background.

4.1 Time evolution

For any non-trivial background, the time-like (frame) vector field (16)

$$e^0 = -i[\mathbf{T}^0, \cdot] \sim \{\mathbf{T}^0, \cdot\} \quad (66)$$

defines a unitary 1-parameter evolution operator

$$U(t) = e^{t[\mathbf{T}^0, \cdot]} \quad (67)$$

on the space of matrices. Since e^0 is a canonical vector field, it respects the natural (Hilbert-Schmidt) inner product on $\text{End}(\mathcal{H}) \sim \mathcal{C}(\mathcal{M})$. Therefore $U(t)$ defines a unitary one-parameter evolution on the space of modes. This evolution is time-like, because the on-shell condition $\square = 0$ can be stated as $e_0^2 = \sum_i e_i^2$. In the semi-classical regime, this defines a time-like flow on spacetime, although the above parameter t is formal and e_0 is typically not normalized w.r.t. the effective metric. For covariant quantum spacetime, $\{t^0, \cdot\} = -\sinh(\tau)\partial_0$ is clearly a time-like vector field on FLRW spacetime, and more generally $e^0|_p = -\frac{\partial}{\partial y^0}$ in local normal coordinates on deformed covariant spacetime $\mathcal{M}^{3,1}$. Even on a background corresponding to a black hole (yet to be found), the evolution $U(t)$ would still be well-defined and unitary.

An arrow of time is also encoded in the $i\varepsilon$ regularization (34), (35) of the action, which induces the Feynman $i\varepsilon$ prescription for 2-point functions $\langle \mathcal{A}(x)\mathcal{A}(y) \rangle$ which are hence time-ordered, incorporating a decomposition into positive and negative energy modes. This is remarkable because the background describes an expanding FLRW universe, with interesting new physics due to the Big Bounce [61, 62].

In particular, this discussion strongly suggests that the time-like matrix $\mathbf{T}^0$ should not be interpreted as time as sometimes advocated [17, 63], but rather as generator of time evolution. Nevertheless, $\mathbf{T}^0 \sim \mathbf{t}^0$ is an interesting function on the underlying symplectic space $\mathcal{M}$, which for covariant quantum spacetime is a 6-dimensional bundle space.

4.2 Physical Hilbert space and $\mathfrak{hs}$ modes

Now consider rigid covariant quantum spacetime (50) as background, which has many physically desirable properties. Its intrinsic higher-spin modes lead however to technical complications, which will be briefly discussed.

The matrix background defines the matrix d'Alembertian $\square \sim \rho^2 \square_G$ which governs the free (quadratic) action for the fluctuations, which is essentially $\mathfrak{hs}$ -valued $\mathcal{N} = 4$ SYM (61). More interesting physics will arise for product backgrounds $\mathcal{M}^{3,1} \times \mathcal{K}$ as discussed below, but for the present purpose it suffices to consider the free action; we always assume or require that these fluctuations are weakly coupled. One can then define the Hilbert space of physical

modes $\mathcal{A}$ as (properly gauge-fixed) space of on-shell modes modulo gauge invariance [64]

$$\mathcal{H}_{\text{phys}} = \{\text{gauge-fixed on-shell modes}\} / \{\text{pure gauge modes}\} \quad (68)$$

This is worked out in full detail in [65], and shown to be a Hilbert space with positive definite norm, without any ghosts. The result is not trivial, because the indices of tangential fluctuations $\mathcal{A}^a$ are contracted with a Minkowski metric, *and* all fields are $\mathfrak{hs}$ valued as discussed below. Nevertheless, the Yang-Mills structure of the action together with the constraints (52) of covariant quantum spacetime ensures that no negative norm states or null states survive in $\mathcal{H}_{\text{phys}}$. A more detailed exposition of this structure can be found in [25, 65], and for a discussion of the tangential modes related to gravity we refer to [66].

$\mathfrak{hs}$ structure of modes. To gain a more detailed understanding, consider the simpler case of transversal fluctuations $T^i \equiv \phi \in \mathcal{C}$ (60), decomposed into $\mathfrak{hs}$ modes

$$\text{End}(\mathcal{H}) = \mathcal{C} = \mathcal{C}^0 \oplus \mathcal{C}^1 \oplus \dots \oplus \dots \quad (69)$$

In the semi-classical limit, the $\mathcal{C}^s$ are modules over $\mathcal{C}^0$, which encode spin s fields on $\mathcal{M}^{3,1}$. Indeed, they can be organized (53)

$$\mathcal{C}^s \ni \phi^{(s)} = \phi_{\mu_1 \dots \mu_s}(x) t^{\mu_1} \dots t^{\mu_s}, \quad \phi_{\mu_1 \dots \mu_s} x^{\mu_i} = 0 \quad (70)$$

as space-like symmetric tensor fields on $\mathcal{M}^{3,1}$. Hence they comprise $2s+1$ degrees of freedom on $\mathcal{M}^{3,1}$, but local Lorentz invariance is not manifest. This is recovered in an alternative characterization of $\mathcal{C}^s$ as space of traceless totally symmetric divergence-free rank s tensor fields on $\mathcal{M}^{3,1}$, via the bijective correspondence [25, 67]

$$\phi^{(s)} \leftrightarrow \tilde{\phi}^{\mu_1 \dots \mu_s}(x) = [\{\dots \{\phi^{(s)}, x^{\mu_1}\}, \dots, x^{\mu_s}\}]_0, \quad \partial_\mu(\rho_M \tilde{\phi}^{\mu_1 \dots \mu_s}) = 0 \quad (71)$$

where $[\cdot]_0$ denotes the projection to $\mathcal{C}^0$. These are no longer space-like, as illustrated by the frame $e^{\dot{\alpha}} = \{\mathbf{T}^{\dot{\alpha}}, x^\mu\}$. Decomposing the modes $\phi \in \mathcal{C}^s$ further into $SO(3,1)$ irreps by imposing an on-shell condition $\square \sim 0$ leads to a non-standard organization because $SO(3,1)$ is the group of space-like isometries, rather than the local Lorentz group¹⁹. These irreps encode massless divergence-free traceless spin s tensors in radiation gauge with 2 degrees of freedom, as well as their derivative descendants; we refer to [25, 65] for details. It seems likely that most of these "massless" irreps recombine into massive modes through quantum effects in the matrix model, but this remains to be clarified in detail.

4.2.1 Deformed covariant spacetime and local normal coordinates

We have seen that the matrix background defines in the semi-classical regime a frame. A general frame is typically $\mathfrak{hs}$ valued²⁰ but always divergence-free [25, 68]

$$e^{\dot{\alpha}\mu} = \{\mathbf{T}^{\dot{\alpha}}, x^\mu\} \in \mathcal{C}, \quad \partial_\mu(\rho_M e^{\dot{\alpha}\mu}) = 0; \quad (72)$$

¹⁹The background does not admit local Lorentz boosts. Local Lorentz invariance is not manifest, but should be recovered as part of the volume-preserving diffeos implemented by gauge transformations.

²⁰This means that the frame is not constant or covariant along the internal S^2 fiber.

the latter is a consequence of the Jacobi identity. Such frames are typically not integrable. It is then not evident how to separate spacetime and $\mathfrak{hs}$ modes, i.e. base and fiber in the 6-dimensional manifold $\mathcal{M}$. That question is addressed in the following.

This question is easy to answer for a special class of frames which respect a subalgebra²¹ $\tilde{\mathcal{C}}^0$ interpreted as functions on (deformed) spacetime $\tilde{\mathcal{M}}^{3,1}$, i.e. $e^a[\tilde{\mathcal{C}}^0] \subset \tilde{\mathcal{C}}^0$. Then frame and torsion tensor (40) are classical i.e. $\tilde{\mathcal{C}}^0$ valued, and describe a 3+1 dimensional spacetime. We denote such backgrounds as *pure backgrounds*. The undeformed background satisfies this condition, and some generalizations are known [25]. However, that property is not generic.

More generally, we focus on backgrounds which are sufficiently mild deformations of (63),

$$\mathbf{T}^{\dot{\alpha}} = \alpha T^{\dot{\alpha}} + \mathcal{A}^{\dot{\alpha}} \quad (73)$$

and consider $\mathcal{A}^{\dot{\alpha}} \in \mathcal{C}$ and the frame $e^{\dot{\alpha}\mu} = \{\mathbf{T}^{\dot{\alpha}}, x^\mu\} \in \mathcal{C}$ as $\mathfrak{hs}$ -valued fields on undeformed spacetime $\mathcal{M}^{3,1}$. Scalar fields and $\mathfrak{hs}$ modes are defined with respect to the undeformed fiber.

In this setting, one can always find **local normal coordinates** (LNC) around any point $p \in \mathcal{M}^{3,1}$, such that the frame is classical (i.e. not $\mathfrak{hs}$ valued) and the metric is covariantly constant at p , as in general relativity. In a sufficiently small neighborhood, any $\mathfrak{hs}$ components of frame, metric and the transition functions are then negligible, since they drop out at the linearized level in the action due to averaging over S^2 . Therefore the local physics on the background is governed by a classical (pseudo-) Riemannian metric and frame. This makes the framework a potentially viable basis for physics.

The precise statement for LNC's is as follows [25, 66]: for any point $p \in \mathcal{M}^{3,1} \subset \mathbb{R}^{3,1}$ and sufficiently mild deformation (73) of the background, one can always choose functions

$$\tilde{y}^\mu : \mathcal{M} \rightarrow \mathbb{R}^{3,1}, \quad \mu = 0, \dots, 3 \quad (74)$$

on $\mathcal{M}$ which vanish on the S^2 fiber over p , such that $\tilde{e}^{\dot{\alpha}\mu} := \{\mathbf{T}^{\dot{\alpha}}, \tilde{y}^\mu\}$ satisfies

$$\begin{aligned} \tilde{e}^{\dot{\alpha}\nu}|_p &= \eta^{\dot{\alpha}\nu}, \\ \partial_\rho \gamma^{\mu\nu}|_p &= 0, \quad \gamma^{\mu\nu} := \eta_{\dot{\alpha}\dot{\beta}} \tilde{e}^{\dot{\alpha}\mu} \tilde{e}^{\dot{\beta}\nu}. \end{aligned} \quad (75)$$

The same statement holds for $G^{\mu\nu}$ replacing $\gamma^{\mu\nu}$ (but not simultaneously). Such $\tilde{y}^\mu$ are denoted as local normal coordinates. They generate a "local" algebra $\tilde{\mathcal{C}}^0$ interpreted as functions on (a local patch of) deformed spacetime $\tilde{\mathcal{M}}^{3,1}$. More technically, they define a deformed bundle projection $\mathcal{M} \rightarrow \tilde{\mathcal{M}}^{3,1}$. Since the S^2 fiber is undeformed at p , the global averaging map $[\cdot]_0 : \mathcal{C} \rightarrow \mathcal{C}^0$ is compatible with the deformed one at p , hence the $\mathfrak{hs}$ modes are undeformed at p .

By construction, this local algebra defines local scalar fields $\tilde{\phi}(\tilde{y}) \in \tilde{\mathcal{C}}^0$, such that $e^{\dot{\alpha}}[\tilde{\phi}]$ has no $\mathfrak{hs}$ components at p . In the vicinity of p , these are the lowest modes among a tower of $\mathfrak{hs}$ modes, just like on undeformed covariant quantum spacetime. The frame $\tilde{e}^{\dot{\alpha}\mu}$ and the metric have arbitrarily small $\mathfrak{hs}$ components in a small neighborhood U of p , and so do the transition functions $\tilde{y}^\mu(\tilde{z})$ for different LNC in nearby patches. We can therefore consistently use the undeformed base space $\mathcal{M}^{3,1}$, find LNC near any points $p \in \mathcal{M}^{3,1}$, and equip $\mathcal{M}^{3,1}$ with a classical frame and metric in LNC around any point p by simply dropping their $\mathfrak{hs}$

²¹this might be interpreted in terms of the frame being constant along a modified bundle map.

components. This allows to compute the length of curves and the distance of different points, leading effectively to a classical (pseudo-) Riemannian metric on $\mathcal{M}^{3,1}$. That procedure is automatically implemented by the matrix model action²² for the lowest fluctuation modes, due to the averaging $[\cdot]_0$.

However, the $\mathfrak{hs}$ components of torsion and curvature are not eliminated in LNC, and they have physical implications in the action which will be briefly discussed in section 7.

Local normal coordinates are worked out in the simple case of dynamical covariant cosmological spacetimes [57, 59], verifying the above picture.

4.3 Gauge transformations and volume-preserving diffeos

Consider gauge transformations $\delta_\Lambda \mathbf{T}^\alpha = \{\Lambda, \mathbf{T}^\alpha\}$ generated by $\Lambda \in \mathcal{C}^1$. Then scalar fields $\phi \in \mathcal{C}^0$ transform under volume-preserving diffeos:

$$\delta_\Lambda \phi = \{\Lambda, \phi\} = \xi^\mu \partial_\mu \phi, \quad \xi^\mu = \{\Lambda, x^\mu\} \quad (76)$$

since $\partial_\mu \{\Lambda, x^\mu\} = 0$ [50]. The frame (and similar tensors such as the torsion) also transforms tensorially, since

$$\delta_\Lambda e^{\dot{\alpha}\mu} \sim \{\Lambda, e^{\dot{\alpha}\mu}\} - \{\mathbf{T}^{\dot{\alpha}}, \xi^\mu\} = \xi^\rho \partial_\rho e^{\dot{\alpha}\mu} - e^{\dot{\alpha}\rho} \partial_\rho \xi^\mu = \mathcal{L}_\xi e^{\dot{\alpha}\mu} \quad (77)$$

using the Jacobi identity. Any volume-preserving vector field ξ^μ on $\mathcal{M}^{3,1}$ can be obtained in this way for suitable $\Lambda \in \mathcal{C}^1$ [25]. Therefore the resulting physics is invariant under (a $\mathfrak{hs}$ extension of) volume-preserving diffeos, which underlies the no-ghost result stated above. In this way, local Lorentz invariance for the classical, tensorial sector of the theory is ensured²³.

5 Compact extra dimensions

The physics resulting from the matrix model on covariant quantum spacetime itself would not be very interesting. To obtain interesting physics, richer backgrounds are required where the 6 transversal scalar fields acquire structure, similar to string theory compactifications. There is indeed a prescription for compactifying the IKKT model on (non)commutative tori by imposing $\mathbf{T}^i \cong \mathbf{T}^i + c^i \mathbb{1}$ up to gauge transformations [69], leading to features familiar from string theory. However, this prescription requires an infinite-dimensional Hilbert space, with infinitely many dof per spacetime volume; then the nonperturbative power of the matrix model is lost.

We therefore *do not* adopt such a procedure, but use a different mechanism to achieve our goal, via **fuzzy extra dimensions**. Despite the name this is an entirely different ansatz, which entails only finitely many dof per volume and thereby maintains the non-perturbative power of the matrix theory. This is achieved by giving non-trivial background to *all* matrices, typically with a product structure $\mathcal{M} \times \mathcal{K}$

$$\langle \mathbf{T}^A \rangle = \begin{pmatrix} \mathbf{T}^{\dot{\alpha}} \otimes \mathbb{1}_{\mathcal{K}} \\ \mathbb{1}_{\mathcal{M}} \otimes \mathcal{K}^i \end{pmatrix}, \quad \dot{\alpha} = 0, 1, 2, 3, \quad i = 4, \dots, 9 \quad (78)$$

²²Therefore it should be consistent with Connes spectral distance for Eucliden covariant quantum spaces.

²³We will see that a non-local sector of gravity does actually break Lorentz invariance.

which can be viewed as product space embedded in target space $\mathbb{R}^{9,1}$. Here $\mathcal{K}$ will typically be some compact fuzzy (i.e. quantized symplectic) space, leading a truncated tower of Kaluza-Klein (KK) modes with scale $m_{\mathcal{K}}^2$ set by the internal matrix Laplacian $\square_6 = [\mathcal{K}^i, [\mathcal{K}_i, \cdot]]$. This can be interpreted as Higgs effect due to SSB of scalar fields in an underlying $U(k)$ gauge theory with background $\mathbf{T}^{\dot{\alpha}} \otimes \mathbb{1}_k$ [70]. Two mechanisms to stabilize $\mathcal{K}$ are known:

1. by imposing an R charge i.e. internal rotation of $\mathcal{K}$ along $\mathcal{M}^{3,1}$ [58, 71, 72], through an ansatz $\mathcal{K}_i^{\pm} = e^{i\varphi(\tau)} T_i^{\pm}$ for 3 complex transversal matrices. Then the kinetic contribution balances the quartic potential of $\mathcal{K}$, leading to an effective potential for T_i with stable non-trivial minimum. The generation of radiation is avoided by imposing the condition

$$\sum_i [\mathcal{K}^{i+}, \mathcal{K}_i^-] \stackrel{!}{=} 0 \quad (79)$$

which ensures that gauge current $J_{\mu} = [\mathbf{T}_i, D_{\mu} \mathbf{T}^i]$ vanishes. This mechanism is appealing because a large R charge stabilizes a large hierarchy between the UV scale on $\mathcal{K}$ and the IR scale of spacetime.

2. for matrix models with extra cubic terms, such as the polarized IKKT model [11]. However the stabilization of a large hierarchy seems doubtful, and maximal SUSY appears to imply the "wrong" mass term for $\mathcal{M}^{3,1}$. Nevertheless, this may be a promising direction to explore; cf. [8–10, 73] for a holographic interpretation of this model.

It is also conceivable that quantum corrections may stabilize $\mathcal{K}$ [74, 75], but this is rather delicate, and the stabilization of a large hierarchy is unclear.

We will therefore adopt the first stabilization mechanism with large R charge. This may have interesting ramifications for the low-energy gauge theory, notably in regard to chirality and CP violation which remain to be understood. There are interesting examples of $\mathcal{K}$ where the lowest KK modes have vanishing R charge and are chiral, being localized at (self-)intersections of $\mathcal{K}$ at the origin of $\mathbb{R}^6$ [71, 76]. These seem to have the low-energy properties required to obtain an effectively Lorentz-invariant low-energy field theory.

6 One-loop effective action and gravity

There is a universal background-independent formula for the 1-loop effective action of the IKKT model. Taking into account the fermions and the gauge-fixing ghosts in the IKKT model, the one-loop effective action²⁴ defined by the Gaussian integral around the background

$$\int_{1\text{ loop}} dT d\Psi d\bar{c} dc e^{iS[T, \Psi, c]} = e^{i(S_0[T] + \Gamma_{1\text{ loop}}[T])} = e^{i\Gamma_{\text{eff}}[T]} \quad (80)$$

²⁴Note that in the context of single-matrix models, the 1-loop contribution is nothing but the Vandermonde determinant. Therefore the 1-loop approach to the multi-matrix model on gauge orbits of matrix backgrounds is analogous to the saddle-point evaluation of the eigenvalue distribution in single matrix models. This is known to give the correct picture, which is completely non-perturbative from a coupling constant point of view.

takes the explicit form [3, 25, 77, 78]

$$\begin{aligned}
\Gamma_{\text{1loop}}[\mathbf{T}] &= \frac{i}{2} \text{Tr} \left(\log(\square - i\varepsilon - \Sigma_{ab}^{(\mathcal{A})}[\mathcal{F}^{ab}, \cdot]) - \frac{1}{2} \log(\square - i\varepsilon - \Sigma_{ab}^{(\psi)}[\mathcal{F}^{ab}, \cdot]) - 2 \log(\square - i\varepsilon) \right) \\
&= \frac{i}{2} \text{Tr} \left(-\frac{1}{4} ((\square - i\varepsilon)^{-1} \Sigma_{ab}^{(\mathcal{A})}[\mathcal{F}^{ab}, \cdot])^4 + \frac{1}{8} ((\square - i\varepsilon)^{-1} \Sigma_{ab}^{(\psi)}[\mathcal{F}^{ab}, \cdot])^4 + \dots \right) \\
&\sim \frac{3i}{4} \int_{\mathcal{M} \times \mathcal{M}} \frac{\Omega_x \Omega_y}{(2\pi)^m} \frac{V_4[\delta\mathcal{F}(x, y)]}{(|x - y|^2 + 2\Delta^2 - i\varepsilon)^4} + \dots
\end{aligned} \tag{81}$$

up to higher-order terms in $\mathcal{F}$. The last expression holds for backgrounds where $\text{End}(\mathcal{H}) \sim \mathcal{C}(\mathcal{M})$ can be interpreted as quantized space of functions on $\mathcal{M}$ (15) with uncertainty scale Δ . Here $\delta\mathcal{F}(x, a) = \mathcal{F}(x) - \mathcal{F}(y)$ and

$$V_4[\delta\mathcal{F}] = -4\text{tr}(\delta\mathcal{F}^4) + (\text{tr}\delta\mathcal{F}^2)^2 \tag{82}$$

where tr denotes the trace over the ab indices with η_{ab} . Several features should be observed:

- The first three terms in the expansion of the log cancel identically due to maximal SUSY, for any background $\mathbf{T}$. Moreover, $\Gamma_{\text{1loop}}[\mathbf{T}]$ is invariant under rescaling $\mathbf{T} \rightarrow c\mathbf{T}$.
- The 1-loop action vanishes for commuting backgrounds and for backgrounds where $\mathcal{F}^{ab}$ is central, such as Moyal-Weyl.
- On a nonabelian Moyal-Weyl background in the Coulomb branch, this effective action is consistent with an expansion of the DBI action for a brane with the 9+1 dimensional $AdS^5 \times S^5$ target space metric [25]. This exhibits the relation with IIB string theory, which is however not the point here.
- The last form in (81) is the bi-local 1-loop effective action *in position space* on $\mathcal{M}$, obtained by computing the trace Tr over the space of all modes using string modes $|x\rangle\langle y|$ [25, 79]. This is of course non-local on $\mathcal{M}$, but it is *weakly non-local* i.e. the locality induced by string modes in the loops is *short-range*, suppressed as $\frac{1}{|x-y|^8}$. This also reflects the relation with IIB supergravity in target space $\mathbb{R}^{9,1}$.
- $V_4[\delta\mathcal{F}] \leq 0$ for Euclidean $\mathcal{F}_{ab}$ of rank ≤ 4 . This may be important to form bounds states in $\mathcal{K}$.

In non-SUSY (or less SUSY) models, the non-local contributions would be long-range, and entail pathological UV/IR divergences at higher loops in 4 dimension. This singles out the IKKT model as essentially unique candidate for a fundamental physical theory²⁵.

6.1 Induced Einstein-Hilbert action

Although the matrix model defines a dynamical quantum geometry, it does not resemble GR unless quantum effects are included. It turns out that the Einstein-Hilbert action arises

²⁵This is analogous to superstring theory, which is singled out among other string theories.

in the 1-loop effective action on quantum spacetime $\mathcal{M}^{3,1}$, but *only* in the presence of $\mathcal{K}$. The reason is that the 1-loop action for $\mathcal{M} \times \mathcal{K}$ includes a mixed term from $V_4[\delta\mathcal{F}]$ which couples the torsions of $\mathcal{M}^{3,1}$ and $\mathcal{K}$, which then reduces to the Einstein-Hilbert action on $\mathcal{M}^{3,1}$. That term in $\Gamma_{1\text{loop}}$ is [79, 80]

$$\Gamma_{1\text{loop}}^{\mathcal{K}-\mathcal{M}} = -\frac{1}{2} \int_{\mathcal{M}} d^4x \frac{\sqrt{G}}{16\pi G_N} T^\rho{}_{\sigma\mu} T_\rho{}^\sigma{}_\nu G^{\mu\nu} = \int d^4x \frac{\sqrt{|G|}}{16\pi G_N} \left(\mathcal{R} + \frac{1}{2} \tilde{T}_\nu \tilde{T}_\mu G^{\mu\nu} - 2\rho^{-2} \partial_\mu \rho \partial^\mu \rho \right) \quad (83)$$

where $\tilde{T}^\mu$ is an axionic vector field defined by the totally antisymmetric torsion, and the Newton constant is determined by the structure of $\mathcal{K}$ as [74]

$$\frac{1}{G_N} \equiv \frac{1}{4\pi\rho^2} \text{Tr}_{\mathcal{K}} \left(\frac{\delta\mathcal{F}_{ij} \delta\mathcal{F}^{ij}}{\square_6} \right) \quad (84)$$

up to higher order terms in $\delta\mathcal{F} = [\mathcal{F}, \cdot]$. The 1-loop contribution of $\mathcal{K}$ to the vacuum energy is [79]

$$\Gamma_{vac,loc}^{\mathcal{K}} = \frac{3i}{4} \text{Tr} \left(\frac{V_{4,\mathcal{K}}}{(\square - i\varepsilon)^4} \right) \sim \int \Omega e^{-\tau} \left(\frac{rm_{\mathcal{K}}}{\alpha} \right)^4 \sum_{\Lambda} \frac{V_{4,\Lambda}}{\mu_{\Lambda}^4} \quad (85)$$

where $\sum_{\Lambda}$ indicates the finite sum over KK modes on $\mathcal{K}$. A sum over $\mathfrak{hs}$ modes truncated by (46) is suppressed in both expressions.

It should be clear that these 1-loop computations are finite and meaningful for two reasons: 1) *maximal SUSY* of the matrix model²⁶, and 2) *finitely many dof* (per volume) of the underlying quantum space. Lack of maximal SUSY would lead to UV divergence and/or a pathological non-locality due to UV/IR mixing, and infinitely dof per volume would lead to divergent internal traces in the above expressions.

The above 1-loop results are obtained for a static $\mathcal{K}$. A non-trivial R charge of $\mathcal{K}$ will have non-trivial effects on the KK spectrum, which remains to be clarified. This may also lead to a selection of acceptable $\mathcal{K}$ backgrounds.

The main message is that gravity arises as quantum effect on quantum spacetime in the IKKT model on suitable backgrounds. This is familiar from string theory, where the present mechanism could be interpreted in terms of 3+1-dimensional branes in a decoupling limit, rather than 9+1-dimensional supergravity in target space. Target space is unphysical in the weakly coupled matrix model regime, thus avoiding the landscape problem.

Propagating gravity modes. The detailed physics of the gravitational sector of the theory is considerably richer than GR, and remains to be understood in detail. To gain some insight, one can elaborate – in the locally flat regime around the reference point $\xi^\mu = (\xi^0, 0, 0, 0) \in \mathcal{M}^{3,1}$ – the gravitational dof. on covariant quantum spacetime following [66]. The $\mathcal{C}^1$ valued perturbations $\mathbf{T}^{\dot{\alpha}} \rightarrow \mathbf{T}^{\dot{\alpha}} = t^{\dot{\alpha}} + \mathcal{A}^{\dot{\alpha}}$ organized as plane waves are

$$\mathcal{A}^{\dot{\alpha}} = A^{\dot{\alpha}\mu}(x) u_\mu, \quad A^{\dot{\alpha}\mu}(x) = A^{\dot{\alpha}\mu} e^{ikx}, \quad A^{\dot{\alpha}0} = 0 \quad (86)$$

²⁶Note that the background is not required to respect SUSY, and of course it should not.

with on-shell condition²⁷ $\square \mathcal{A}^{\dot{\alpha}} = 0 = k_{\mu} k^{\mu}$ after gauge-fixing $\partial_{\dot{\alpha}} A^{\dot{\alpha}\mu} = 0 = k_{\dot{\alpha}} A^{\dot{\alpha}\mu}$. Subtracting the 3 residual on-shell pure gauge modes $k^{\dot{\alpha}} \Lambda^{\mu}$ for $\mu = 1, 2, 3$, we obtain $6 = 4 + 1 + 1$ physical on-shell modes, which might be viewed as 4 modes of a partially massive graviton, a conformal (dilaton) model and an axion. Specifically, for light-like

$$k^{\dot{\alpha}} = (k, k, 0, 0) \equiv k^+ \quad (87)$$

the 6 physical modes are

$$\mathcal{H}_{phys} = \{A^{2\mu}, A^{3\mu}\} \quad (88)$$

for space-like $\mu = 1, 2, 3$, which satisfy the space-like condition

$$\xi_{\dot{\alpha}} A^{\dot{\alpha}\mu} = 0. \quad (89)$$

We can separate these into 4 transversal modes $A_{(tran)}^{\dot{\alpha}\mu} k_{\mu} = 0$ and 2 longitudinal modes $A_{(long)}^{\dot{\alpha}1}$, which will lead to extra $\mu = +$ frame modes (93).

Recall that $\mathcal{A}$ is a *potential* for the frame. The effective frame modes are obtained as

$$\delta e^{\dot{\alpha}\nu} = \{A^{\dot{\alpha}\mu} u_{\mu}, x^{\nu}\} \propto A^{\dot{\alpha}\mu} u_{\mu} k_{\sigma} (\xi^{\sigma} u^{\nu} - \xi^{\nu} u^{\sigma}) \quad (90)$$

which is divergence-free

$$\partial_{\nu} \delta e^{\dot{\alpha}\nu} = 0 \quad (91)$$

and transversal space-like in $\dot{\alpha}$ due to the Lorentz gauge condition for A and

$$\xi_{\dot{\alpha}} \delta e^{\dot{\alpha}\mu} = 0 = \partial_{\dot{\alpha}} \delta e^{\dot{\alpha}\nu}. \quad (92)$$

However they are no longer space-like in ν . The 6 physical frame modes are given by

$$\{\delta e^{2\mu}, \delta e^{3\mu}\} \quad \text{for } \mu = +, 2, 3 \quad (93)$$

which contain components in $\mathcal{C}^0$ and $\mathcal{C}^2$. The classical $\mathcal{C}^0$ components of $A_{(tran)}^{\dot{\alpha}\mu}$ lead to 4 transversal frame modes with $\mu = 2, 3$, while the two $\mu = +$ modes arising from $A_{(long)}^{\dot{\alpha}1}$ are longitudinal $[\delta e_{(long)}^{\dot{\alpha}\mu}]_0 \propto w^{\dot{\alpha}} k^{\mu}$. The latter lead to diffeo-like helicity ± 1 graviton modes but are nevertheless physical frame modes, while the former lead to 2 transversal graviton modes and a trace mode, as the gauge group consists of volume-preserving diffeos only. All graviton modes satisfy the harmonic gauge condition $\partial_{\mu} h^{\mu\nu} = 0$. The remaining physical frame mode is captured by the axion, and possibly in the $\mathcal{C}^2$ components of the graviton.

²⁷This applies to the Yang-Mills action and to the 1-loop effective action at quadratic order. However their combination leads to extra modes with modified dispersion relation discussed in section 7.

6.2 Induced mass terms and massless modes

Quantum fluctuations of the background lead to induced mass terms²⁸ $\text{Tr}(\mathcal{A}^\dot{\alpha}\mathcal{A}_\dot{\alpha})$ in the 1-loop effective action, as shown by explicit computations [75]. This is important on covariant quantum spaces, because it removes the phletora of $\mathfrak{hs}$ modes from the low-energy physics, notably the massless $\mathfrak{hs}$ -valued nonabelian fields. On the other hand it is puzzling, since mass terms for gauge fields are usually incompatible with gauge transformations

$$\delta_\Lambda \mathcal{A}^\dot{\alpha} = [\mathcal{A}^\dot{\alpha}, \Lambda] + [\mathbf{T}^\dot{\alpha}, \Lambda] . \quad (94)$$

For noncommutative backgrounds however, mass terms *can* be compatible with gauge transformations for the following reason: A quadratic term in the action of the form

$$\text{Tr}(\mathbf{T}^\dot{\alpha} + \mathcal{A}^\dot{\alpha})(\mathbf{T}_\dot{\alpha} + \mathcal{A}_\dot{\alpha}) = \text{const} + 2\text{Tr}(\mathbf{T}^\dot{\alpha}\mathcal{A}_\dot{\alpha}) + \text{Tr}(\mathcal{A}^\dot{\alpha}\mathcal{A}_\dot{\alpha}) \quad (95)$$

is clearly invariant under gauge transformations

$$\mathbf{T}^\dot{\alpha} + \mathcal{A}^\dot{\alpha} \rightarrow U^{-1}(\mathbf{T}^\dot{\alpha} + \mathcal{A}^\dot{\alpha})U. \quad (96)$$

We can check explicitly that $\text{Tr}(\mathcal{A}\mathcal{A}) + 2\text{Tr}(\mathbf{T}\mathcal{A})$ is gauge invariant:

$$\delta_\Lambda \text{Tr}(\mathcal{A}_\dot{\alpha}\mathcal{A}^\dot{\alpha}) = 2\text{Tr}(\mathcal{A}_\dot{\alpha}[\mathcal{A}^\dot{\alpha}, \Lambda]) + 2\text{Tr}(\mathcal{A}_\dot{\alpha}[\mathbf{T}^\dot{\alpha}, \Lambda]) = 2\text{Tr}(\mathcal{A}_\dot{\alpha}[\mathbf{T}^\dot{\alpha}, \Lambda]) \quad (97)$$

which precisely cancels with

$$\delta_\Lambda 2\text{Tr}(\mathbf{T}_\dot{\alpha}\mathcal{A}^\dot{\alpha}) = 2\text{Tr}(\mathbf{T}_\dot{\alpha}[\mathcal{A}^\dot{\alpha}, \Lambda]) + 2\text{Tr}(\mathbf{T}_\dot{\alpha}[\mathbf{T}^\dot{\alpha}, \Lambda]) = 2\text{Tr}(\mathbf{T}_\dot{\alpha}[\mathcal{A}^\dot{\alpha}, \Lambda]) . \quad (98)$$

This mechanism is particularly transparent in the ARS model, where adding a mass term will modify the radius of the vacuum sphere $\mathbf{T}^a \rightarrow \alpha \mathbf{T}^a$, while tangential fluctuations around the new background remain massless. The same applies to covariant quantum spacetime in the IKKT model, which in fact requires an induced mass term (or vacuum energy) for stabilization, as discussed above. Moreover, we can offer an argument why the $\mathcal{C}^1$ -valued gravity modes and the $\mathcal{C}^0$ -valued unbroken nonabelian gauge fields are protected from quantum effects in the presence of $\mathcal{K}$ and remain massless:

Indeed the $\mathcal{C}^1$ -valued deformations $\mathcal{A}^\dot{\alpha} = \delta\Lambda^{\dot{\alpha}}_{\dot{\beta}}(x)\mathbf{T}^{\dot{\beta}}$ for x -dependent $SO(3,1)$ generators $\delta\Lambda^{\dot{\alpha}}_{\dot{\beta}}(x)$ can be considered as 5 Goldstone bosons of the global $SO(3,1)$ symmetry²⁹, which guarantees that these are flat directions of the quantum effective potential. In the presence of non-trivial $\mathcal{K}$, the same argument for the global $SO(9,1)$ symmetry implies that nonabelian gauge fields $\mathcal{A}^\dot{\alpha} \in \mathcal{C}^0$ in the commutant of $\mathcal{K}$ (i.e in the unbroken gauge group) are massless; recall that all other YM modes acquire large KK masses via the Higgs mechanism.

For $\mathfrak{hs}$ -valued fluctuations $\mathcal{A}^\dot{\alpha} \in \mathcal{C}^s$ with $s > 1$, the mass term $\text{Tr}(\mathcal{A}_\dot{\alpha}\mathcal{A}^\dot{\alpha})$ is separately gauge invariant, since the mixed term $\text{Tr}(\mathbf{T}_\dot{\alpha}\mathcal{A}^\dot{\alpha})$ vanishes identically. The protection mechanism via Goldstone modes does not apply to these, since no VEV exists for spin $s > 1$. Therefore these are expected to acquire genuine (large) mass through quantum effects.

Finally, we recall that graviton are derived quantities, and mass terms for those are forbidden as usual since they transform as $\delta h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu}$ under volume-preserving diffeos.

²⁸The $\mathfrak{hs}$ modes are massless at the classical level because the background frame generates a rank 4 effective metric on $\mathcal{M} = \mathbb{C}P^{1,2}$. Adding quantum fluctuations to the background will contribute kinetic energy along the fiber, acting like an effectively 6-dimensional metric on $\mathcal{M}$, leading to mass terms for the $\mathfrak{hs}$ modes.

²⁹Note that the $SO(3,1)$ orbit of $\{t^a\}$ over $\mathcal{M}^{3,1}$ is 5-dimensional, and this argument does not apply to the dilaton.

7 Large-scale modifications of gravity & mirage matter

In the matrix model, the fundamental degrees of freedom are matrices and their fluctuations $\mathcal{A}$, which – on a non-trivial background – play the role of *potentials* for the frame (16). This leads to important differences from the traditional formulation and mechanism of gravity. In particular, the semi-classical matrix action is on Yang-Mills type, and *cannot* be rewritten as a local action for the frame. This entails to some rather unfamiliar IR modifications of gravity. The combination of YM and EH terms also leads to extra dof due to the higher-derivative structure in terms of $\mathcal{A}$.

At one loop, the gravity sector of the effective geometrical gauge theory contains (at least) the following three types of terms

$$S_{\text{grav}} = S_{\text{EH}} + S_{\text{YM}} + S_{\text{vac}} \quad (99)$$

where S_{EH} is the Einstein-Hilbert term (extended by dilaton and axion) (83), S_{YM} is the semi-classical Yang-Mills type matrix action, and S_{vac} is the induced vacuum energy. These can be computed explicitly at weak coupling. Let us focus on the YM term:

$$S_{\text{YM}} = - \int d^4x \sqrt{G} \rho^{-2} \mathcal{F}^{\dot{\alpha}\dot{\beta}} \mathcal{F}_{\dot{\alpha}\dot{\beta}}, \quad \mathcal{F}_{\dot{\alpha}\dot{\beta}} = -\{\mathbf{T}_{\dot{\alpha}}, \mathbf{T}_{\dot{\beta}}\} \quad (100)$$

This cannot be written as a local function of the frame. To cast the equations of motion into the form of generalized Einstein equations, we rewrite the variation of this term as

$$\delta S_{\text{YM}} = -4 \int d^4x \sqrt{G} \rho^{-2} \delta e^{\dot{\alpha}\mu} C_{\dot{\alpha}\mu}, \quad \delta e^{\dot{\alpha}\mu} = \{\delta \mathbf{T}^{\dot{\alpha}}, x^\mu\} \quad (101)$$

where the "anharmonicity tensor" $C_{\dot{\alpha}\mu}$ [81] incorporates the background flux via $\{C_{\dot{\alpha}\mu}, x^\mu\} = -e^{\dot{\beta}}[\mathcal{F}_{\dot{\beta}\dot{\alpha}}]$ up to an ambiguity $C_{\dot{\alpha}\mu} \rightarrow C_{\dot{\alpha}\mu} + \partial_\mu C_{\dot{\alpha}}$. An approximate solution is

$$C_{\dot{\alpha}\mu} \approx -r^{-4} \alpha^2 \Box^{-1} \{e^{\dot{\beta}}[\mathcal{F}_{\dot{\beta}\dot{\alpha}}], x^{\mu'}\} \gamma_{\mu\mu'} \quad (102)$$

where $\Box := r^{-4} \alpha^2 \{x^\mu, \gamma_{\mu\nu} \{x^\nu, \cdot\}\} \sim \frac{1}{3} \alpha^2 e^\tau \Box_{G,c \rightarrow \frac{1}{\sqrt{3}}c}$ is a modified d'Alembertian with speed of light reduced by $\frac{1}{\sqrt{3}}$, including also $\mathfrak{h}\mathfrak{s}$ contributions which we drop. The variation of the YM term can then be cast into the geometric form

$$\begin{aligned} \delta S_{\text{YM}} &= -4 \int d^4x \rho_M C_{\dot{\alpha}\nu} \{\delta \mathbf{T}^{\dot{\alpha}}, x^\nu\} \\ &\equiv - \int d^4x \sqrt{G} \rho^{-2} (T_{\mu\nu}[C] - \tfrac{1}{2} G_{\mu\nu} T[C] + B_{\mu\nu}[C]) e_{\dot{\alpha}}^\mu \delta e^{\dot{\alpha}\nu}. \end{aligned} \quad (103)$$

Here $T_{\mu\nu}[C]$ is an effective energy-momentum tensor, interpreted as "mirage" matter since it arises as a "non-local reflection" of ordinary matter $T_{\mu\nu}$ as shown below. Dropping the antisymmetric term $B_{\mu\nu}[C]$, we thus obtain modified Einstein equations³⁰ [66, 81]

$$\frac{1}{8\pi G_N} \mathcal{G}_{\mu\nu} = T_{\mu\nu} + T_{\mu\nu}[C] - G_{\mu\nu} \tilde{\Lambda}. \quad (104)$$

³⁰Note that the divergence constraint (91) of the frame is taken care of by the above ambiguity of $C_{\dot{\alpha}\mu}$, which is used to make $T_{\mu\nu}[C]$ divergence-free.

At the linearized level, the mirage energy momentum tensor can be written explicitly using (102) as

$$\delta T^{\mu\nu}[C] \approx -2\alpha^2 r^{-4} \Box^{-1} \Box_G \delta \bar{G}^{\mu\nu} \quad (105)$$

where $\delta \bar{G}^{\mu\nu} = \delta G^{\mu\nu} - \frac{1}{2} G^{\mu\nu} (G \delta G)$ is the trace-reversed metric perturbation. Then the linearized Einstein equation $\delta \mathcal{G}^{\mu\nu} = 8\pi G_N (T^{\mu\nu} + \delta T^{\mu\nu}[C])$ takes the form

$$(1 - m^2 \Box^{-1}) \delta \mathcal{G}^{\mu\nu} \approx 8\pi G_N T^{\mu\nu} \quad (106)$$

in harmonic gauge, in terms of the scale³¹

$$m^2 := 32\pi G_N \alpha^2 r^{-4} \quad (107)$$

which marks the cross-over between the GR and YM regime. This can be seen to be an IR scale [66], which depends on the background under consideration. An equivalent form is

$$\delta \mathcal{G}^{\mu\nu} \approx 8\pi G_N (T^{\mu\nu} + \delta T^{\mu\nu}[C]), \quad \delta T^{\mu\nu}[C] := \frac{m^2}{\Box - m^2} T^{\mu\nu} \quad (108)$$

which exhibits the origin of mirage matter as a non-local reflection of ordinary matter. A tentative fully non-linear form of these equations was also given in [66]

$$\frac{1}{8\pi G_N} \mathcal{G}^{\mu\nu} = T^{\mu\nu} + T^{\mu\nu}[C] - G^{\mu\nu} \tilde{\Lambda} \approx \frac{1}{m^2} \Box T^{\mu\nu}[C] \quad (109)$$

but this should be taken with a (big) grain of salt, since all these quantities are $\hbar\mathfrak{s}$ valued. The action $S_{\text{EH}} + S_{\text{YM}}$ including $\hbar\mathfrak{s}$ components to quadratic order is worked out in [66].

The resulting modifications of GR are not well understood up to now. However, the most obvious and interesting conclusion is that gravity is modified in the IR in a non-local manner, distinct from any conventional modifications of GR. For scales shorter than m^{-1} , we recover (linearized) GR, since $\delta T^{\mu\nu}[C] \approx 0$ for $\Box \gg m^2$. For longer distances, a screening behavior due to mirage matter is found, with $\delta T_{\mu\nu}[C](k) \rightarrow -T_{\mu\nu}(k)$ in the IR limit $k \rightarrow 0$. In particular, linearized (!) gravity effectively has a finite range, because the Yang-Mills action clearly dominates the EH action in the extreme IR, which has 2 extra derivatives. Then the geometry is governed by the classical matrix model plus vacuum energy, leading to a cosmology which is far less sensitive to matter than in GR. This leads us back to cosmological quantum spacetime $\mathcal{M}^{3,1}$, which is not governed by the Friedmann equations and does not seem to require any fine-tuning. There is no need to invoke inflation to explain the homogeneity of the observed universe, and trans-Planckian issues [82] are naturally resolved by the matrix framework.

It is tempting to conjecture that these IR modifications of gravity are (at least partially) responsible for so-called dark energy and dark matter. Some simple calculations of the IR modifications at the linearized level are given in [66], where halo-like effects are indeed observed. However it is premature to make any quantitative claim at this point. The validity of the linearized approximation is very limited, and a different behavior may arise at the

³¹This scale is in Cartesian local normal coordinates, see [66] for a more careful discussion.

non-linear level. A safe statement is that the gravitational sector contains an extra "mirage" sector which goes beyond GR, leading to some apparently non-local dynamics if viewed in the language of GR.

It should be mentioned that the combination of EH and YM terms leads to 4-derivative equations of motion for $\mathcal{A}$ [66]. They admit massless modes with standard dispersion $\square_G = 0$, hence gravitational waves should propagate as in GR. However, there are extra modes propagating at a reduced speed, with apparent tachyonic mass of order $-m^2$. These seem to be unstable at face value, however this is expected to be an artifact, and a possible stabilization mechanism based on vacuum energy was identified [25]. These modes can also be seen in (106) ff., where vacuum energy contributions were dropped. This brings us back to the question of stability of the background, which is of course essential, but presumably resolved by vacuum energy as the induced E-H term becomes irrelevant []. It may also be useful in this context to recall that the matrix model admits a conserved matrix energy momentum tensor $[\mathbf{T}_a, \mathcal{T}^{ab}] = 0$, which should keep potential instabilities in bound.

8 Conclusion and outlook

We have described how near-realistic quantum spacetime and gravity can arise from the IKKT matrix models within a weak-coupling approach. The basic hypotheses of the underlying scenario can now be justified to some degree, including candidate backgrounds describing a 3+1-dimensional FLRW spacetime, stabilization of fuzzy extra dimensions, and the emergence of an Einstein-Hilbert term. The semi-classical effective frame and metric with Minkowski signature are completely understood in the weak coupling regime, leading to an understanding of gravity as a quantum effect on quantum spacetime.

In particular, covariant quantum spacetime provides a serious candidate for spacetime, leading to a consistent and appealing physical picture. While we focus on the $k = -1$ case, similar backgrounds with $k = 0$ [57] and $k = +1$ [41] are also possible. Some understanding why $\mathfrak{hs}$ modes acquire a mass is offered in section 6.2, so that (near-) realistic physics seems feasible on such a background.

There are many questions and tasks which remain to be addressed. Clearly the most important problem is a detailed understanding of the gravitational dynamics. We have seen that this is highly non-trivial, since metric and frame are not fundamental but derived objects, which are moreover $\mathfrak{hs}$ -valued. The concept of local normal coordinates discussed in section 4.2.1 is crucial to eliminate these $\mathfrak{hs}$ components and make the connection to physics, which needs to be implemented and understood more generally. Furthermore, the "mirage matter" arising from the Yang-Mills flux and its physical significance needs to be understood better at the non-linear level. This is closely related to the crossover scale m (107), whose actual size depends on the background but needs better understanding. Finally, stability of the cosmic background is an important issue, which will require taking into account vacuum energy effects. This in turn is important to understand the dynamics of the cosmic evolution and its cosmic fluctuations. These and many other questions clearly require much more work, which is hopefully sparked and facilitated by the present review.

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