Quantum Geometry: The Spectral Approach

Lecture I: The algebraic vision of (Noncommutative) Geometry

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Why Geometry?

Geometry is tied closely to physics. Galilei wrote in the Saggiatore (1623).

Philosophy is written in this great book which is continually open before our eyes—I mean the universe—but it cannot be understood unless one first learns to understand the language and to recognize the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering in vain through a dark labyrinth.

La filosofia è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi a gli occhi (io dico l'universo), ma non si può intendere se prima non s'impara a intender la lingua, e conoscer i caratteri, ne' quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche [. . .] senza questi è un aggirarsi vanamente per un oscuro laberinto.

Several physical theories do not just use geometry as a tool, they are geometry

Classical (analytic) mechanics is nothing but the symplectic geometry of phase space.

General relativity has taught us that gravity is the Riemannian geometry of curved spacetime.

But the geometric view of mechanics had a crisis with the advent of the quantum world with the uncertainty principle

All at once the notion of a point in phase space becomes untenable, and the geometry made of points need to be generalised to describe the quantum phase space

The algebra of commuting functions on phase space becomes an algebra quantum noncommuting operators.

Why Quantum Spacetime?

The dynamical variable of general relativity is spacetime itself, encoded, in the metric $\boxed{g^{\mu\nu}}$, or in derived quantities: Riemann tensor, Christoffel symbols.

It is natural then to think of $\boxed{g}$ as a field and quantize it using the usual procedures of quantum field theory.

I am sure you that this does work. The resulting theory is not UV complete renormalizable. It works as effective theory, but in the ultraviolet an infinte number of counterterms are need .

On the other side there are important conceptual differences between $\boxed{g^{\mu\nu}(x)}$ and a regular field. The latter is defined on. background spacetime, with its topology, metric and all structures. Not to speak of the enormous difficulties related to diffeomorphism invariance!

The idea is then to quantize spacetime itself. Taking inspiration from the quantization of classical phase space

There are more motivations to introduce some sort of “pointless” spaces. Without any ambition to be complete I may mention, in chronological order:

1936 Bronstein: concentrating enough energy to localize the gravitational field itself disturbs the geometry, creating what later will be called a black hole.

1947 Snyder, and later Yang, consider noncommuting coordinates, in the hope of solving the problems of infinities in quantum field theory.

1964 Mead uses a “gravitational” version of the Heisenberg microscope to argue that there should be a uncertainty in the measure of positions of the order of the Planck length squared.

1994 Doplicher, Fredenhagen and Roberts: to avoid gravitational collapse needs spacetime uncertainty relations and noncommuting coordinates.

1999 Seiberg and Witten find that in some limit of string theory, coordinates do not commute.

In general all of these paper do not cite the previous ones, except SW who cite Snyder, but not Yang.

Noncommutative Geometry

Hence we need of a generalization of geometry, along the lines of what happens for quantum mechanics. This was first advocated by Von Neumann, who called it pointless geometry.

One of the main characteristic of quantization, giving it the name, is the fact that often quantities are discretized, although there is continuous spectrum as well. The spectrum is a key word for quantum spacetime as well.

I will describe the spectral approach to quantum spacetime, mainly due to Alain Connes. Although the natural scale is Planckian, surprisingly there is an interpretation of the standard model of particle interaction (in a background gravitational field) as a very simple noncommutative geometry. Spacetime is still "classical", and the noncommutativity is given by a matrix algebra. The description has the possibility to be confronted with experiments and to make predictions. I will mention something in the following, but there is no time for a complete treatment.

From Commutative to Commutative geometry

Hausdorff topological spaces as commutative algebras

There is an important series of theorems, mostly due to Gelfand, Naimark and Segal, which connect $\boxed{C^*}$-algebras and Hausdorff topological spaces.

A $\boxed{C^*}$-algebra $\boxed{\mathcal{A}}$ is an associative, normed algebra over a field (typically the complex $\boxed{\mathbb{C}}$), with an involution $\boxed{*}$ which satisfies the following properties:

$$\boxed{\|a+b\| \leq \|a\| + \|b\| \;\;,\;\; \|\alpha a\| = |\alpha|\,\|a\| \;\;,\;\; \|ab\| \leq \|a\|\;\|b\|}$$

$$\boxed{\|a\| \geq 0, \quad \|a\| = 0 \iff a = 0\;, \|a^*\| = \|a\| \;\;, \quad \|a^* a\| = \|a\|^2 \;\;, \quad \forall\; a \in \mathcal{A}}$$

A $\boxed{C^*}$-algebra is complete in the topology given by the norm, otherwise we talk of a $\boxed{*}$-algebra.

Examples

- The algebra of $\boxed{n \times n}$ matrices. The $\boxed{*}$ is Hermitean conjugations. One has to be careful with the norm. The obvious norm $\boxed{\mathrm{Tr}\, A^\dagger A, \;\; A \in \mathrm{Mat}(n, \mathbb{C})}$ does not work. The proper $\boxed{C^*}$-norm is:

$$\boxed{\|A\|^2 = \max_{\text{eigenvalues}} A^\dagger A}$$

- The algebra $\boxed{\mathcal{C}_0(M)}$ of continuous functions over a Hausdorff spaces $\boxed{M}$ (vanishing on the frontier in the noncompact case), with pointwise multiplication. Also in this case the obvious $\boxed{L^2}$ norm $\boxed{\| \;\; \|_2}$

$$\boxed{\|f\|_2^2 = \int \mathrm{d}x |f(x)|^2}$$

does not work. The proper norm in this case is

$$\boxed{\|f\|^2 = \sup_{x \in M} |f(x)|^2}$$

Ex Find, for the previous cases, examples for which $\boxed{\|f^* f\| \neq \|f\|^2}$ for the "wrong" norm.

Note also that with the $\boxed{\text{sup}}$ norm the algebra of continuous functions is complete, while with the $\boxed{\| \; \|_2}$ norm the completion leads to the $\boxed{L^2}$ (with the usual Borel measure). Note that $\boxed{L^2}$ is not an algebra, summing two $\boxed{L^2}$ does not lead necessarily to a square summable function.

If we were to start with more restricted spaces, for example compact support or rapid descend, the completion will lead to continuous functions anyway.

If instead I start from $\boxed{L^2 \cap L^\infty}$, which is an algebra, and complete with the sup norm (again with not too fancy measure) I get measurable functions.

In particular the second case is relevant. Given a Hausdorff space $\boxed{M}$ it is always possible to define canonically a commutative $\boxed{C^*}$-algebra: continuous complex valued functions. Only if $\boxed{M}$ is compact the algebra will be unital (will contain the identity).

A key result is that the inverse is also true:

A commutative $\boxed{C^*}$-algebra always is the algebra of continuous complex valued functions on a Hausdorff space

In other words, Hausdorff topological spaces and commutative $\boxed{C^*}$-algebra are in a one-to-one correspondence

The proof is constructive, given a $\boxed{C^*}$-algebra it is possible to reconstruct the points of the Hausdorff space, and their topology

For this we need the notion of state $\boxed{\rho}$, i.e. a linear functional $\boxed{\rho : \mathcal{A} \to \mathbb{C}}$ positive $\boxed{\rho(a^*a) \geq 0}$ and of norm one $\boxed{\|\rho\| = \sup_{\|a\| \leq 1} \rho(a) = 1}$

If the algebra is unital then it must be $\boxed{\rho(1) = 1}$

The space of states is convex, any linear combination of states of the kind $\boxed{\lambda\rho_1 + (1-\lambda)\rho_2}$, for $\boxed{\lambda \in [0,1]}$, is still a state.

Some states cannot be expressed as such convex sum, they form the boundary of the set and are called pure states.

Ex Find the states (and the pure ones) for the algebra of $\boxed{n \times n}$ matrices.

For a commutative algebra the pure states coincide with the (necessarily one-dimensional) irreducible representations, as well as the set of maximal and prime ideals. In the noncommutative case these sets are different.

Gelfand and Naimark gave a prescription to reconstruct a topological space in an unique way from a commutative algebra

The topology on the space of pure states is given by the limit. Given a succession of pure states $\boxed{\delta_{x_n}}$ define the limit to be

$$\boxed{\lim_n \delta_{x_n} = \delta_x \Leftrightarrow \lim_n \delta_{x_n}(a) = \delta_x(a) \ , \forall a \in \mathcal{A}}$$

In other words the states which correspond to the points are the evaluation maps whereby the complex number associated to a function is simply the value of the function in the point

With the above topology the starting algebra results automatically the algebra of continuous functions over the space of states

Similarly it is possible to give a topology on the set of ideals, leading to the same topology for the commutative case.

Two examples

Apart from $\boxed{\mathbb{C}}$ itself, the simplest case is $\boxed{\mathbb{C}^n}$, simply vectors $\boxed{\vec{A}}$ of complex numbers added, and multiplied component by component.

A state is a map from these vectors into complex:

$$\boxed{\rho(\vec{A}) = \sum_i c_i A_i}$$

the requirements of of positivity and unitality require $\boxed{c_i}$ real and

$$\boxed{\sum_i c_i = 1, \ , \ c_i > 0}$$

A generic state can be written as a sum of two other states. Given $\boxed{\{c_1, c_2, \ldots\}}$, take $\boxed{\lambda = c_1}$, then

$$\boxed{\{c_1, c_2, \ldots\} = \lambda\{1, 0, \ldots\} + (1-\lambda)\left\{0, \frac{c_2}{1-c_1}, \ldots\right\}}$$

Note that the second expression is a state since $\boxed{\sum_{i=2}^{n} c_i = 1 - c_1}$.

If $\boxed{c_1 = 0}$ it is possible to do the same for any other nonzero $\boxed{c_i}$.

The construction will not work for states of the kind $\boxed{\delta_i(\vec{A}) = A_i}$, corresponding a sets of $\boxed{c}$'s which are all zero except one.

In this case it is impossible to find, for $\boxed{j \neq i}$ two nonzero positive numbers for which $\boxed{\lambda c'_j + (1-\lambda)c''_j = 0}$.

We can conclude that there are $\boxed{n}$ pure states corresponding to the $\boxed{\delta_i}$ defined above.

Coherently with what I said earlier the correspond to the irreducible representation of the algebra $\boxed{\pi_i(\vec{A}) = A_i}$, so that $\boxed{\pi_i(\vec{A} + \vec{B}) = \pi_i(\vec{A}) + \pi_i(\vec{B})}$ and $\boxed{\pi_i(\vec{A}\,\vec{B}) = \pi_i(\vec{A})\,\pi_i(\vec{B})}$.

Therefore to this algebra correspond a collection of $\boxed{n}$ points.

Any convergent sequence must associate, after some steps, the same value to $\boxed{i}$, therefore the topological space is a collection of singletons, equipped with the discrete topology, all points open and closed.

The construction works also when $\boxed{n}$ is infinite, in this case we ket an infinite lattice.

In the previous cases the algebras where either finite dimensional, or had a countable basis, and we got topological states with the discrete topology.

Therefore if have a vector space, a rule on how to multiply the vectors, an involution and a proper norm, it is always possible to find a corresponding topological space.

In the infinite dimensional case it is difficult to give these data in an abstract way. Reverse the process, start from the continuous functions on $\boxed{\mathbb{R}}$ *vanishing at infinity*, and show that the construction reproduces the real line.

The norm is given by the absolute value of the maximum of the function. The identity is not part of the algebra.

A state is a map from the functions to complex numbers, these can be obtained integrating with a positive definite, normalised density function. You may see it as a measure $\boxed{\rho}$ such that $\boxed{\rho(x) \geq 0 \ \forall x}$ and $\boxed{\int \mathrm{d}x \rho(x) = 1}$.

$$\boxed{\rho(f) = \int_{-\infty}^{\infty} \mathrm{d}x \, \rho(x) f(x)}$$

Just as we did earlier one can write the state as a sum of two states. Just take any $\boxed{\rho_i}$ such that $\boxed{\lambda\rho_1(x) + (1-\lambda)\rho_2(x) = \rho(x)}$.

These states are distributions on the test space of continuous functions vanishing at infinity. But there are distributions which cannot be expressed by functions: Dirac's $\boxed{\delta}$.

$$\boxed{\delta_{x_0}(f) = \int \mathrm{d}x \delta(x - x_0) f(x) = f(x_0)}$$

In way physicists speak, there are not two "functions" such that $\boxed{\lambda\rho_1(x) + (1-\lambda)\rho_2(x) = \delta(x - x_0)}$

Clearly there are as many of these states as there are point of $\boxed{\mathbb{R}}$.

Also in this case the pure states are in a one to one correspondence with the irreducible representations of the algebra. And they also correspond to the maximal ideals (ideals which are not contained in any other ideal, apart from the whole algebra itself.

Ex Prove that the maximal ideals of this algebra correspond to the points. Hint: functions which vanish on a subset of $\boxed{\mathbb{R}}$ form an ideal.

Topology is obvious. Remember we are cheating and we started from continuous functions, namely such that

$$\boxed{\lim_n x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)}$$

Since $\boxed{\delta_{x_n}(f) = f(x_n)}$ the proof is straightforward.

Ex Removing the requirement that function vanish at infinity, and requiring just that they have a definite limit gives another algebra. To which topological space it corresponds?

Algebras as operators: the GNS construction

These lectures were advertised as presenting the spectral point of view. The reason is that any $\boxed{C^*}$-algebra is always representable as bounded operators on a Hilbert space. The proof is again constructive, and is called the Gelfand-Naimark-Segal (GNS) construction.

Every algebra is to star with a vector space, and it has an action on itself. As first step consider this vector space as starting point for construction of the Hilbert space. We then need an inner product with certain properties, and then to complete in the norm given by this product. This norm will not be the norm of the $\boxed{C^*}$-algebra.

Start with a state (any state) $\boxed{\rho}$. Then define a bilinear map

$$\boxed{\langle a, b \rangle = \rho(a^* b)}$$

This map has almost the correct properties of the inner product.

The exception is that there may be some elements of the algebra for which $\boxed{\rho(a^* a) = 0}$, even if $\boxed{a}$ is not the null vector. Those states form a (left) ideal $\boxed{\mathcal{N}_\rho}$, namely if $\boxed{a \in \mathcal{N}_\rho}$ then $\boxed{ab \in \mathcal{N}_\rho \ \forall b \in \mathcal{A}}$.

Ex Prove that $\boxed{\mathcal{N}_\rho}$ is an ideal

Then we can quotient out $\boxed{\mathcal{N}_\rho}$, i.e. consider the vector space $\boxed{\mathcal{A}/\mathcal{N}_\rho}$ of the equivalence classes $\boxed{[a]}$ of all elements which differ from $\boxed{a}$ by an element of $\boxed{\mathcal{N}_\rho}$.

Now the scalar product

$$\boxed{\langle [a], [b] \rangle = \rho(a^* b)}$$

is well defined since it is easy to see that is independents on the representatives $\boxed{a}$ and $\boxed{b}$ in the equivalence classes

We have this given a scalar product which defines a good norm. The Hilbert space is defined completing in the scalar product norm, on which $\boxed{\mathcal{A}}$ acts in a natural way as bounded operators.

Ex Perform the GNS construction for $\boxed{\mathrm{Mat}(n, \mathbb{C})}$ starting from a pure state.

Ex Given the algebra $\boxed{\mathcal{C}_0(\mathbb{R})}$ consider the two states $\boxed{\delta_{x_0}(f) = f(x_0)}$ and $\boxed{\rho(a) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx\, e^{-x^2} a(x)}$. Find the Hilbert space in the two cases.

Noncommutative Spaces. Morita equivalence

Once established the correspondence between Hausdorff spaces and commutative $\boxed{C^*}$-algebras we may ask what happens for noncommutative algebras

It is still possible to give topologies on the spaces of irreducible representations (not anymore necessarily one-dimensional) , pure states and maximal or primitive ideals, but these do not coincide anymore.

For us a noncomutative space will be, by extension of the concept, a noncommutative $\boxed{C^*}$-algebra, sometimes it will possible to talk of point, possibly generalized, other times it simply does not make sense.

Think for example of the algebra of quantum operators generated by $\boxed{\widehat{p}}$ and $\boxed{\widehat{q}}$ of quantum mechanics. This is a deformation of the ordinary algebra of functions on phase space. Pure states are square integrable functions on $\boxed{\mathbb{R}}$ which are in no correspondence with the original points. Coherent states are the closest to the concept of point, but are in no way "pointlike".

Consider instead the case of matrix valued functions on $\boxed{\mathbb{R}}$. The algebra is noncommutative and obviously there is an underlying space, $\boxed{\mathbb{R}}$ itself!

This fact is captured by the concept of Morita Equivalence

We first need the concept of Hilbert Module. This is a generalization of Hilbert spaces where the field $\boxed{\mathbb{C}}$ is replaced by a $\boxed{C^*}$-algebra. A right Hilbert module $\boxed{\mathcal{E}}$ over $\boxed{\mathcal{A}}$ is a right module equipped with a sesqulinear form $\boxed{\mathcal{E} \times \mathcal{E} \to \mathcal{A}}$ linear in the first variable and with

$$\boxed{\langle \eta_1 |\, \eta_2 a \rangle_{\mathcal{A}} = \langle \eta_1 |\, \eta_2 \rangle_{\mathcal{A}}\, a \ , \ \langle \eta_1 |\, \eta_2 \rangle_{\mathcal{A}}^* = \langle \eta_2 |\, \eta_1 \rangle_{\mathcal{A}} \ ,}$$

$$\boxed{\langle \eta |\, \eta \rangle_{\mathcal{A}} \geq 0 \ , \quad \langle \eta |\, \eta \rangle_{\mathcal{A}} = 0 \ \Leftrightarrow \ \eta = 0}$$

$\boxed{\forall \eta_1, \eta_2, \eta \in \mathcal{E},\ a \in \mathcal{A}}$

Completion in the norm given $\boxed{\|\eta\|_{\mathcal{A}}^2 = \|\, \langle \eta |\, \eta \rangle \,\|_{\mathcal{A}}}$ is assumed.

A left Hilbert module is defined in an analogous way.

Ex Make $\boxed{\mathcal{A}^N}$ into a Hilbert module, and discuss its automorphisms in the cases $\boxed{\mathcal{A}}$ is $\boxed{\mathrm{Mat}(n, \mathbb{C})}$ or $\boxed{C_0(M)}$.

Two $\boxed{C^*}$-algebras $\boxed{\mathcal{A}}$ and $\boxed{\mathcal{B}}$ are said to be Morita equivalent if there exists a full right Hilbert $\boxed{\mathcal{A}}$-module $\boxed{\mathcal{E}}$ which is at the same time a left $\boxed{\mathcal{B}}$-module in such a way that the structures are compatible

$$\boxed{\langle\eta|\,\xi\rangle_{\mathcal{B}}\,\zeta = \eta\,\langle\xi|\,\zeta\rangle_{\mathcal{A}}\ , \qquad \forall\ \eta,\xi,\zeta\in\mathcal{E}}$$

Ex The algebras $\boxed{\mathbb{C},\mathrm{Mat}(\mathbb{C},n)}$ and and $\boxed{\mathcal{K}}$, compact operators on a Hilbert space are all Morita equivalent. Find the respective bimodules

Two Morita equivalent algebras have the same space of representations, with the same topology. They describe the same noncommutative space.

The space of complex valued, or matrix valued functions, since a matrix algebra has only one representation, are Morita equivalent.

But Morita equivalence is far from being "physical" equivalence. The states for example are different. There is more than topology!

Beyond topology: metric aspects

So far the ingredients were a $\boxed{C^*}$-algebra $\boxed{\mathcal{A}}$, acting as bounded operators on $\boxed{\mathcal{H}}$. I will now briefly describe how the usual elements of geometry can be introduced. This is the Noncommutative Geometry programmed championed by Alain Connes, who introduced the term in the early eighties.

The crucial addition is a self-adjoint operator $\boxed{D}$ on $\boxed{\mathcal{H}}$. It is not necessarily bounded, but with compact resolvent. It can be seen as a generalization of the Dirac operator. I will call it Dirac operator, even of oftentimes it will not look at all as the operator introduced by Paul Dirac.

Together $\boxed{\mathcal{A}, D, \mathcal{H}}$ form what is called a spectral triple

The Dirac operator enables to describe, in purely algebraic operatorial terms, the usual structures of geometry. Since the description is purely algebraic it can easily be generalised to the noncomutative case, thus enabling a non-commutative geometry.

The presence of $\boxed{D}$ enables, for example to give a distance, and hence a metric structure, on the space of states (not necessarily pure) on an algebra

$$\boxed{d(\rho_1,\rho_2) = \sup_{\|[D,a]\|\leq 1} \{|\rho_1(a) - \rho_2(a)|\}}$$

Ex Take $\boxed{\mathcal{A} = \mathcal{C}(\mathbb{R})}$ and $\boxed{D = i\partial_x}$. Prove that the distance among pure states gives the usual distance among points of the line $\boxed{d(x_1,x_2) = |x_1 - x_2|}$

The original Dirac operator $\boxed{\gamma^\mu\partial_\mu}$ "knows" a lot about a spin manifold. The differential structure,the spin structure, but also the metric tensor since $\boxed{\{\gamma^\mu,\gamma^\nu\} = g^{\mu\nu}}$, it is the square root of the laplacian. It is therefore no surprise that it plays such an important role in pure mathematics.

This distance reproduces the distance defined by $\boxed{g^{\mu\nu}}$ for the cases of ordinary manifolds.

$\boxed{D}$ enables a purely algebraic definition of one-forms and the creation of a cohomology via $\boxed{da \sim [D, a]}$. A generic one form will be $\boxed{A = \sum_i a_i [D, b_i]}$

The construction is delicate, one has to implement $\boxed{d^2 a = 0}$. To achieve this one has to quotient out some "junk" forms.

The number of dimensions of the space can be obtained from the rate of growth of the eigenvalues of $\boxed{D^2}$ (Weyl). Consider the ratio of number $\boxed{N_\omega}$ of eigenvalues smaller than a value $\boxed{\omega}$. Then

$$\boxed{\lim_{\omega \to \infty} \frac{N_\omega}{\omega^{\frac{d}{2}}}}$$

For $\boxed{d}$ too small this value diverges, for $\boxed{d}$ large diverges. There is only one value for which this number is finite, this defines the dimension, and is of course the usual Hausdorff dimension for the usual cases.

Ex Verify that the square, the sphere and the disc have dimension $\boxed{2}$.

Integrals are substituted by trace, in particular a regularized sum of eigenvalues called the Diximier trace

The programme is the construction of a Dictionary, whereby the concepts of differential geometry are translated in a purely algebraic way.

In this way they can be generalized to the cases in which the algebra is noncommutative, or even if they are not deformation of ordinary geometries.

There is an algebraic definition of manifolds, for which two other operators are needed: generalized chirality operator $\boxed{\Gamma}$, present only in even dimensions, which splits $\boxed{\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R}$, and a "charge conjugation" antiunitary operator $\boxed{J}$ which gives a real structure.

It is connected to the Tomita-Takesaki operator, in case you know what this is

Connes' NCG and the standard model

I will say just a few words on how the standard model of particles interaction emerges from this approach to noncommutative geometry. I will be extremely sketchy, more details in the references.

The first step is to allow fluctuations of the $\boxed{D}$ by one forms, i.e. potentials

$$\boxed{D_A = D + \sum_i a_i [D, b_i] = D + A}$$

Ex The construction is made for compact Euclidean spaces. Find two good reasons for which this construction cannot be performed, without changes, for noncompact or Minkowkian spacetimes.

Ex Consider the toy model for which $\boxed{\mathcal{A}}$ is $\boxed{\mathbb{C}^2}$ represented as diagonal matrices on $\boxed{\mathcal{H} = \mathbb{C}^m \oplus \mathbb{C}^n}$, and $\boxed{D = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix}}$. Find $\boxed{D_A}$

Introduce now the Spectral Action whose boonic part is:

$$\boxed{S_B = \mathrm{Tr}\,\chi\left(\frac{D_A}{\Lambda}\right)}$$

$\boxed{\chi}$ is a cutoff functions, like the characteristic function of the interval $\boxed{[0,1]}$, or some smoothened version of it, and $\boxed{\Lambda}$ is a cutoff scale.

This action must be read in a renormalization scheme. Namely we must insert it in a path integral and read the action coming from it. And expanded with heath kernel techniques

The fermionic part (to be regularised is

$$\boxed{S_F = \langle\Psi|D_A\Psi\rangle}$$

I am skipping all details regarding charge conjugation and chirality, although they are crucial..

If we apply this to the usual manifold, with $\boxed{D = \gamma^\mu \partial_\mu}$ we obtain the QED action.

If instead we apply this procedure to the algebra which is the tensor product of the previous algebra times a finite dimensional $\boxed{\mathcal{A}_F}$ the product first three smallest $\boxed{C^*}$-algebras:

$$\boxed{\mathcal{A}_F = \mathrm{Mat}(3, \mathbb{C}) \oplus \mathbb{H} \oplus \mathbb{C}}$$

$\boxed{3 \times 3}$ matrices, quaternions, seen as $\boxed{2 \times 2}$ matrices, and complex numbers.

The unitary elements of this algebra is $\boxed{U(3) \times SU(2) \times U(1)}$. Almost the gauge group of the standard mode, but the extra $\boxed{U(1)}$ can be taken care with the imposition of an extra unimodularity condition.

As Hilbert space we take the product of Dirac spinors times a finite dimensional space of $\boxed{96}$ component vectors comprising all kwon fermions: two quarks by three colours, plus two leptons, times two for antiparticles, times two for chirality, times three for generation.

The Dirac operator has the continuous part, and a part acting on the finite part. A $\boxed{96 \times 96}$ whose non zero elements connect left/right fermions/antifermions by a real number, the Yukawa coupling, which are related to the mass of the fermions, and are experimentally determined.

Then one cranks the machine, using heath kernel and renormalization. I am skipping details which took years of work!

If you want to know the final result . . .

$$
\begin{gathered}
\mathcal{L}_{SM} = -\frac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc}f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \\
\frac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \frac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - igc_w(\partial_\nu Z^0_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu \partial_\nu W^-_\mu - \\
W^-_\mu \partial_\nu W^+_\mu) + Z^0_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - igs_w(\partial_\nu A_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu(W^+_\mu \partial_\nu W^-_\mu - \\
W^-_\mu \partial_\nu W^+_\mu) + A_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - \frac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \frac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + \\
g^2 c_w^2 (Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2 (A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w (A_\mu Z^0_\nu (W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \frac{1}{2}\partial_\mu H \partial_\mu H - 2M^2\alpha_h H^2 - \partial_\mu\phi^+\partial_\mu\phi^- - \frac{1}{2}\partial_\mu\phi^0\partial_\mu\phi^0 - \\
\beta_h\left(\frac{2M^2}{g^2} + \frac{2M}{g}H + \frac{1}{2}(H^2+\phi^0\phi^0+2\phi^+\phi^-)\right) + \frac{2M^4}{g^2}\alpha_h - g\alpha_h M\left(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-\right) - \\
\frac{1}{8}g^2\alpha_h\left(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2\right) - gMW^+_\mu W^-_\mu H - \\
\frac{1}{2}g\frac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \frac{1}{2}ig\left(W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)\right) + \\
\frac{1}{2}g\left(W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) + W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)\right) + \frac{1}{2}g\frac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) + \\
M\,(\frac{1}{c_w}Z^0_\mu\partial_\mu\phi^0 + W^+_\mu\partial_\mu\phi^- + W^-_\mu\partial_\mu\phi^+) - ig\frac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_wMA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) - \\
ig\frac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_wA_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \frac{1}{4}g^2W^+_\mu W^-_\mu\left(H^2+(\phi^0)^2+2\phi^+\phi^-\right) - \\
\frac{1}{8}g^2\frac{1}{c_w^2}Z^0_\mu Z^0_\mu\left(H^2+(\phi^0)^2+2(2s_w^2-1)^2\phi^+\phi^-\right) - \frac{1}{2}g^2\frac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) - \\
\frac{1}{2}ig^2\frac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \frac{1}{2}g^2s_wA_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) + \frac{1}{2}ig^2s_wA_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - \\
g^2\frac{s_w}{c_w}(2c_w^2-1)Z^0_\mu A_\mu\phi^+\phi^- - g^2s_w^2A_\mu A_\mu\phi^+\phi^- + \frac{1}{2}ig_s\,\lambda^a_{ij}(\bar q^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu - \bar e^\lambda(\gamma\partial + m^\lambda_e)e^\lambda - \bar\nu^\lambda(\gamma\partial + \\
m^\lambda_\nu)\nu^\lambda - \bar u^\lambda_j(\gamma\partial + m^\lambda_u)u^\lambda_j - \bar d^\lambda_j(\gamma\partial + m^\lambda_d)d^\lambda_j + igs_wA_\mu\left(-(\bar e^\lambda\gamma^\mu e^\lambda) + \frac{2}{3}(\bar u^\lambda_j\gamma^\mu u^\lambda_j) - \frac{1}{3}(\bar d^\lambda_j\gamma^\mu d^\lambda_j)\right) + \\
\frac{ig}{4c_w}Z^0_\mu\{(\bar\nu^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar e^\lambda\gamma^\mu(4s_w^2-1-\gamma^5)e^\lambda) + (\bar d^\lambda_j\gamma^\mu(\tfrac{4}{3}s_w^2-1-\gamma^5)d^\lambda_j) + (\bar u^\lambda_j\gamma^\mu(1-\tfrac{8}{3}s_w^2 + \\
\gamma^5)u^\lambda_j)\} + \frac{ig}{2\sqrt2}W^+_\mu\left((\bar\nu^\lambda\gamma^\mu(1+\gamma^5)U^{lep}{}_{\lambda\kappa}e^\kappa) + (\bar u^\lambda_j\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d^\kappa_j)\right) + \\
\frac{ig}{2\sqrt2}W^-_\mu\left((\bar e^\kappa U^{lep}{}^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar d^\kappa_j C^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)u^\lambda_j)\right) + \\
\frac{ig}{2M\sqrt2}\phi^+\left(-m^\kappa_e(\bar\nu^\lambda U^{lep}{}_{\lambda\kappa}(1-\gamma^5)e^\kappa) + m^\lambda_\nu(\bar\nu^\lambda U^{lep}{}_{\lambda\kappa}(1+\gamma^5)e^\kappa)\right) + \\
\frac{ig}{2M\sqrt2}\phi^-\left(m^\lambda_e(\bar e^\lambda U^{lep}{}^\dagger_{\lambda\kappa}(1+\gamma^5)\nu^\kappa) - m^\kappa_\nu(\bar e^\lambda U^{lep}{}^\dagger_{\lambda\kappa}(1-\gamma^5)\nu^\kappa)\right) - \frac{g}{2}\frac{m^\lambda_\nu}{M}H(\bar\nu^\lambda\nu^\lambda) - \frac{g}{2}\frac{m^\lambda_e}{M}H(\bar e^\lambda e^\lambda) + \\
\frac{ig}{2}\frac{m^\lambda_\nu}{M}\phi^0(\bar\nu^\lambda\gamma^5\nu^\lambda) - \frac{ig}{2}\frac{m^\lambda_e}{M}\phi^0(\bar e^\lambda\gamma^5 e^\lambda) - \frac{1}{4}\,\bar\nu_\lambda\,M^R_{\lambda\kappa}\,(1-\gamma_5)\hat\nu_\kappa - \frac{1}{4}\,\overline{\bar\nu_\lambda\,M^R_{\lambda\kappa}\,(1-\gamma_5)\hat\nu_\kappa} + \\
\frac{ig}{2M\sqrt2}\phi^+\left(-m^\kappa_d(\bar u^\lambda_j C_{\lambda\kappa}(1-\gamma^5)d^\kappa_j) + m^\lambda_u(\bar u^\lambda_j C_{\lambda\kappa}(1+\gamma^5)d^\kappa_j)\right) + \\
\frac{ig}{2M\sqrt2}\phi^-\left(m^\lambda_d(\bar d^\lambda_j C^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m^\kappa_u(\bar d^\lambda_j C^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j)\right) - \frac{g}{2}\frac{m^\lambda_u}{M}H(\bar u^\lambda_j u^\lambda_j) - \frac{g}{2}\frac{m^\lambda_d}{M}H(\bar d^\lambda_j d^\lambda_j) + \\
\frac{ig}{2}\frac{m^\lambda_u}{M}\phi^0(\bar u^\lambda_j\gamma^5 u^\lambda_j) - \frac{ig}{2}\frac{m^\lambda_d}{M}\phi^0(\bar d^\lambda_j\gamma^5 d^\lambda_j)
\end{gathered}
$$

Here the notation is as in [46], as follows.

- Gauge bosons: $A_\mu, W^\pm_\mu, Z^0_\mu, g^a_\mu$
- Quarks: u^κ_j, d^κ_j, collective : q^σ_j
- Leptons: e^λ, ν^λ
- Higgs fields: $H, \phi^0, \phi^+, \phi^-$
- Ghosts: G^a, X^0, X^+, X^-, Y,
- Masses: $m^\lambda_d, m^\lambda_u, m^\lambda_e, m_h, M$ (the latter is the mass of the W)
- Coupling constants $g = \sqrt{4\pi\alpha}$ (fine structure), $g_s =$ strong, $\alpha_h = \frac{m_h^2}{4M^2}$
- Tadpole Constant β_h
- Cosine and sine of the weak mixing angle c_w, s_w
- Cabibbo–Kobayashi–Maskawa mixing matrix: $C_{\lambda\kappa}$
- Structure constants of SU(3): f^{abc}
- The Gauge is the Feynman gauge.

And this is only the gauge part, there is also the part due to the nontrivial metric.

In the end we find the Lagrnagian of the standard model coupled to a back-groung gravitational field, with the the Higgs potential, the Einstein-Hilbert Lagrangian correctd by terms quadratic in the Riemann tensor.

No wonder in this, we had chosen he proper noncommutative geometry, with the $\boxed{\gamma}$ matrices in a curved background and put all of tha parameters of the standard model in the finite part of the Dirac operator

Except one...

The parameters of the Higgs potential, which in the end give the Higgs mass, are not an input, they come out as functions of all the other parameters, which are dominated by the top Yukawa coupling (mass).

Therefore it is possible to predict the mass of the Higgs.

The calculation gives the value of 170 GeV.

Which is wrong. The correct one is 125 GeV.

At this point there can be two kind of attitudes. One the one side one may say: “The number is wrong. The model has been falsified.”

On the other side one may consider a success that a model, based on pure mathematical considerations, get a number which is in the right ball park. The mass of the Higgs could have been of the order of the mass of the electron, or the QCD scale.

Tue model can be slightly altered, making is similar to a Pati-Salam model, with is a grand unified group is $\boxed{SU(2)_L \times SU(2)_R \times SU(4)}$. But there is loss of predictivity, rather than a sharp prediction there is a window of values, and the prediction of an extra boson.

The field is still active, but I will not pursue it in these lectures.

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