Quantum Geometry: The Spectral Approach

Lecture II: Field Theories

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The most famous Noncommutative Space: Grönewold-Moyal product

Actually the most famous noncommutative space is the phase space of a quantum particle. It is described by two noncommuting variables, $\boxed{p}$ and $\boxed{q}$, whose commutator is a constant. Indeed most of the techniques we will use today come from quantum mechanics

A great impulse to noncommutative geometry came without doubt from a paper by Seiberg and Witten who found that, in a particular limit of string theory, one could have that:

$$\boxed{[x^\mu, x^\nu] = \theta^{\mu\nu}}$$

Where $\boxed{\theta^{\mu\nu}}$ is a constant parameter.

SW also introduced field theory on this noncommutatative space. This is done via a noncommutative deformation of the algebra of functions.

The idea, which goes back to quantum mechanics, is to deform the usual, commutative product among functions, with the introduction of a $\boxed{\star}$ non-commutative product, such that:

$$\boxed{x^\mu \star x^\nu - x^\nu \star x^\mu = [x^\mu, x^\nu]_\star = \mathrm{i}\theta^{\mu\nu}}$$

The matrix $\boxed{\theta}$ is antisymmetric, with a different coordinate choice it is possible to make it block diagonal, with $\boxed{2 \times 2}$ blocks, and effectively work in two dimensions.

The generalisation to an arbitrary number of dimensions is notationally cumbersome, and not totally straightforward. But it can be done, with some care. I will also be in Euclidean framework, Wick rotation is possible, but again, to be handled with care. Moreover, you may wonder what happens to Lorentz symmetry. You are right to worry, if I have time I will comment on this.

We all know Dirac's procedure to associate operators to noncommuting variables. With the present notations I may represent $\boxed{\widehat{x}^1}$ on the functions of x^1 by multiplicative operators, and $\boxed{\widehat{x}^2}$ by $\boxed{-\mathrm{i}\theta\partial_{x^1}}$.

To simplify notations set $\boxed{\theta = 1}$, in case it may be reinstated from dimensional considerations.

But which operator is associated to objects like $\boxed{x^1 x^2}$, or $\boxed{\mathrm{e}^{-x^{1^2}-x^{2^2}}}$? There are ordering ambiguities.

These are solved by the Weyl map, which associates to an ordinary function an operator.

The inverse Wigner map associates functions to operator. Let me introduce them with some generality.

Start with an operator $\boxed{\hat{A}}$ acting on a (finite or infinte dimensional) $\boxed{\mathcal{H}}$.

Introduce two distinct operators, called "quantizer" and "dequantizer" $\boxed{\hat{U}(x)}$ and $\boxed{\hat{D}(x)}$. These depend on a set of parameters* $\boxed{x = (x_1, \ldots, x_N)}$ which in our case we interpret as coordinates of the space we want quantize.

The two operators must satisfy $\boxed{\text{tr}\left(\hat{D}(x')\hat{U}(x)\right) = \delta(x' - x)}$. With a Kronecker $\boxed{\delta}$ for the finite dimensional case.

The dequantization map associates to the operator the function (which may actually be a distribution) $\boxed{f_A(x)}$:

$$\boxed{f_A = \Omega^{-1}\left(\hat{A}\right) \equiv \text{tr}\left(\hat{A}\hat{U}(x)\right).}$$

*We use subscripts rather than superscript to avoid confusion with exponents. The metric is Euclidean.

The quantization map is the inverse

$$\boxed{\hat{A} = \hat{\Omega}(f_A) = \int \mathsf{d}x f_A(x) \hat{D}(x)}$$

Ex Verify that the two maps are one the inverse of the other.

The original Weyl-Wigner maps are given by:

$$\boxed{\hat{D}_{\mathsf{M}}(x) = \frac{1}{(2\pi)^2} \hat{U}_{\mathsf{M}}(x) = \int \frac{\mathsf{d}^2\xi}{(2\pi)^2} \exp\left[\mathsf{i}\left(\xi_1(\hat{x}_2 - x_2) + \xi_2(\hat{x}_1 - x_1)\right)\right]}$$

The Weyl quantization map is therefore:

$$\boxed{\hat{\Omega}(f) = \hat{f} = \int \mathrm{d}^2 x f(x) \int \frac{\mathrm{d}^2\xi}{(2\pi)^2} \exp\left[\mathrm{i}\left(\xi_1(\hat{x}_2 - x_2) + \xi_2(\hat{x}_1 - x_1)\right)\right]}$$

Note that the map corresponds to make a Fourier transform of the function, and then antitrasform with $\boxed{\hat{x}^1}$ rather then the usual variables.

Ex Find the operator corresponding to the function $\boxed{x_1^n x_2^m}$.

The inverse map is:

$$\boxed{f(x) = \Omega^{-1}(\hat{f}) = \operatorname{tr}\left(\hat{f} \int \mathrm{d}^2\xi \exp\left[\mathrm{i}\left(\xi_1(\hat{x}_2 - x_2) + \xi_2(\hat{x}_1 - x_1)\right)\right]\right).}$$

The Weyl map associates Hermitean operators to real functions. It maps $\boxed{L^2}$ functions into Hilbert-Schmidt operators, Schwartzian functions into trace class operators.

We are now ready to define a $\boxed{\star}$ product. The steps are the following:

1. Take two functions $\boxed{f_A(x),\ f_B(x)}$.
2. Associate the two operators $\boxed{\hat{f}_A = \hat{\Omega}(f_A)}$ and $\boxed{\hat{f}_B = \hat{\Omega}(f_B)}$
3. Multiply The two operators.
4. Use $\boxed{\Omega^{-1}}$ to obtain again a function.

$$\boxed{(f_A \star f_B)(x) = \Omega^{-1}(\hat{\Omega}(f_A)\hat{\Omega}(f_B))}$$

The product is is in general noncommutative, but it is always associative.

In the spirit of the previous lecture I am constructing a noncommutative product to describe a noncommutative geometry, and I am doing it with operators. This gives a justification to the use of matrix models.

I know, the mathematician would jump at me since the equivalence operators/matrices is wrong, or naive if they want to be generous. There are issues of boundedness, domains, nuances in the spectral decompositions. . . Nevertheless this approach gives solid ways to do computations, as we see in the other lectures, to make an example.

I will now present a matrix base for the $\boxed{\star}$ product. I will also connect to coherent states, which as we know are the best approximation to points for a wide class of noncommutative spaces. I will reinstate $\boxed{\theta}$ because it is useful to see the limits.

Go form the $\boxed{x}$ coordinates to a complex plane coordinates, with a slightly different convention:

$$\boxed{z = \frac{1}{\sqrt{2}}(x_1 + \mathrm{i}x_2).}$$

Define creations and annihilation operators in the usual way:

$$\boxed{\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{x}_1 - \mathrm{i}\hat{x}_2), \qquad \hat{a} = \frac{1}{\sqrt{2}}(\hat{x}_1 + \mathrm{i}\hat{x}_2), \qquad [a, a^\dagger] = \theta,}$$

The eigenfuctions of $\boxed{\hat{a}}$ are the usual coherent states $\boxed{\hat{a}|z\rangle = z|z\rangle}$.

The decomposition of the identity reads $\boxed{\mathbf{1} = \frac{1}{\pi\theta}\int \mathrm{d}^2z|z\rangle\langle z|.}$ Coherent states are non-orthogonal $\boxed{\langle z|z'\rangle = \mathrm{e}^{-\frac{1}{\theta}(\bar{z}z + \bar{z}'z' - 2\bar{z}z')}.}$

Introduce the usual number operator $\boxed{\hat{N} = \hat{a}^\dagger \hat{a}}$ and its eigenvalues and eigen-vectors $\boxed{\hat{N}\,|n\rangle = n\theta\,|n\rangle}$. We have that:

$$\boxed{\hat{a}|n\rangle = \sqrt{n\theta}|n-1\rangle, \qquad \hat{a}^\dagger|n\rangle = \sqrt{(n+1)\theta}|n+1\rangle, \qquad |n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!\theta^n}}|0\rangle}$$

The quantizer is simple to define in this basis

$$\boxed{\hat{D}(z) = \int \mathrm{d}^2\xi \mathrm{e}^{\frac{\theta}{2}(\xi(a^\dagger - \bar{z}) - \bar{\xi}(a-z))}}$$

The Moyal product can be given the popular form

$$\boxed{(f \star g)(z,\bar{z}) = f(z,\bar{z}) \exp\left[\frac{\theta}{2}\left(\overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}} - \overleftarrow{\partial_{\bar{z}}}\overrightarrow{\partial_z}\right)\right] g(z,\bar{z}),}$$

where $\boxed{\overleftarrow{\partial}}$ (resp. $\boxed{\overrightarrow{\partial}}$) acts on the left (resp. on the right).

The Moyal product is not the only $\boxed{\star}$ product, it is not even the only product which gives $\boxed{x^\mu \star x^\nu - x^\nu \star x^\mu = \mathrm{i}\theta^{\mu\nu}}$. We will see another one in the following.

There is a general theory of deformation of products. It was developed for quantum mechanics, ed in fact one problem was to find products which, to first order, reproduce a given Poisson bracket.

The problem is associativity. It was a challenging problem for the community, with partial results, until Kontsevich solved it completely in 1997.

Field theory deformed by the star product has been studied intensely, for example consider the deformed $\boxed{\phi^4}$ theory described by the action:

$$\boxed{S = \int \partial_\mu \phi(x) \star \partial_\mu \phi(x) + \frac{m^2}{2} \phi(x) \star \phi(x) + \frac{\lambda}{4!} \phi(x)^{\star 4}}$$

.

For the Moyal product it results that $\boxed{\int f \star g = \int f g}$, and therefore the quadratic part, and hence the free propagator is unchanged.

What instead is substantially changed is the vertex, which, in momentum space, becomes

$$\boxed{S_{\mathrm{int}} = \frac{\lambda}{4!} \int \prod_{i=1}^{4} \frac{d^d p_i}{(2\pi)^d} (2\pi)^d \delta^{(d)}\left(\sum_{i=1}^{4} p_i\right) \prod_{i=1}^{4} \tilde{\phi}(p_i) \exp\left[-\frac{\mathrm{i}}{2} \sum_{i<j} p_i \wedge p_j\right],}$$

with $\boxed{p \wedge q = p_\mu \theta^{\mu\nu} q_\nu.}$

A consequence of this vertex is that planar and nonplanar diagrams behave differently. For planar diagrams there is no change, for non planar there is a softening of the ultraviolet divergences (the original dream of Snyder).

The wild oscillations of $\boxed{\exp\left[-\frac{\mathrm{i}}{2}\sum_{i<j} p_i \wedge p_j\right]}$ act as a regulator.

The bad news is that when integrating ia a loop with momentum $\boxed{k}$, then the fase, for a given channel has $\boxed{k \wedge (p_i + p_j)}$.

When $\boxed{p_i + p_j \to 0}$ the exponent in the loop vanishes, and there is a infrared divergence. This is called Ultraviolet/Infrared (UV/IR) mixing.

Field theories with a variety of products are a very active field of research. I will mention the Grosse-Wulkenhhar model, renormalizabel to all orders, which uses the matrix basis we introduce next in crucial way. The actions is

$$\boxed{S[\phi] = \int d^4x \left[\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi) \star (\tilde{x}^\mu\phi) + \frac{m^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi^{\star 4}\right],}$$

with $\boxed{\widetilde{x}_\mu = 2(\theta^{-1})_{\mu\nu}x^\nu.}$

The model is renormalizable to all orders, it has a built in UV/IR symmetry.

The Moyal product, because of the presence of $\boxed{\theta}$, breaks Lorentz invariance. The point is that:

A quantum spacetime may require a quantum symmerty.

This leads to the theory of quantum groups, another notable omission in these lectures.

In some cases, notably $\boxed{\kappa}$-Minkowsky, the symmetry precedes the noncommutative geometry described by the commutator

$$\boxed{[x^0, x^i] = \frac{\mathrm{i}}{\kappa} x^i, \qquad [x^i, x^j] = 0.}$$

I have no time to discuss quantum groups, but I leave you with a teaser: A Lie group is a manifold, and the parameters of the transformations are coordinate functions, what would happen if the do not commute?

Back to Weyl quatization. Introduce the shift operator:

$$\boxed{\widehat{\mathcal{D}}(z,\bar{z}) = \mathrm{e}^{\frac{1}{\theta}(z\hat{a}^\dagger - \bar{z}\hat{a})}, \qquad \widehat{\mathcal{D}}(z,\bar{z})|w\rangle = |w+z\rangle \mathrm{e}^{\frac{1}{2\theta}(z\bar{w}-w\bar{z})}.}$$

Then (skipping some passages):

$$\boxed{\widehat{U}(z) = 2\widehat{\mathcal{D}}(z)\,(-1)^{\widehat{N}}\,\widehat{\mathcal{D}}(-z) = 2\,(-1)^{(\hat{a}^\dagger-\bar{z})(\hat{a}-z)}\,,}$$

This expression enables us to express the Weyl quantization of functions in the discrete basis of the number operator in an efficient way, thus effectively associate a matrix (with the caveats above) to any function.

the basis for these operators is given by the $\boxed{|n\rangle\langle m|}$, seen as operator. Operators in this base multiply as matrices, given $\boxed{F = \sum_{nm} f_{nm}\,|n\rangle\langle m|\,,\ \ G = \sum_{nm} g_{nm}\,|n\rangle\langle m|\,,}$

$$\boxed{FG = H = \sum_{nm} h_{nm}\,|n\rangle\langle m|\,,\ \ h_{nm} = \sum_p f_{np}g_{pm}}$$

All sums are $\boxed{0}$ to $\boxed{\infty}$.

The procedure is the following:

- Express the functions as a Taylor series

$$\boxed{f = \sum_{mn} f_{nm} \bar{z}^n z^m}$$

- Weyl quantize:

$$\boxed{\hat{\Omega}\left(\bar{z}^m z^n\right) = \sum_{k=0}^{\min(m,n)} \frac{k!}{2^k} \binom{m}{k} \binom{n}{k} \left(\hat{a}^\dagger\right)^{m-k} \hat{a}^{n-k}.}$$

Ex Calculate $\boxed{\hat{\Omega}\left(\bar{z}^m z^n\right)}$.

- Then we use

$$\boxed{(\hat{a}^\dagger)^r \hat{a}^s = \sum_{q=s}^{\infty} \frac{\sqrt{q!(q-s+r)!}}{(q-s)!} |q-s+r\rangle\langle q|}$$

Ex Calculate this expression. Hint: $\boxed{\hat{a} = \sum_{n=0}^{\infty} \sqrt{n+1}\, |n\rangle\langle n+1|}$

Finally the expression becomes (reinstating $\boxed{\theta}$)

$$\boxed{\hat{\Omega}(f) = \sum_{m,n=0}^{\infty} f_{mn}\, \theta^{\frac{m+n}{2}} \sum_{k=0}^{\min(m,n)} \frac{k!}{2^k} \binom{m}{k} \binom{n}{k}}$$

$$\boxed{\times \sum_{q=m-k}^{\infty} \frac{\sqrt{q!\,(q+n-m)!}}{(q-m+k)!} |q+n-m\rangle\langle q|.}$$

This expression is not very easy to use, and is some cases it may be easier to calculate $\boxed{\hat{\Omega}(f)}$ using for example the quantizer, and express the operator in the matrix base simply calculating its matrix elements $\boxed{\langle m|\,\hat{\Omega}(f)\,|n\rangle}$.

This is what we do next with the very interesting example, that of a normalized Gaussian centred in the origin (again without $\boxed{\theta}$)

$$\boxed{\rho = \frac{\lambda}{\pi} \mathrm{e}^{-\lambda(x_1^2+x_2^2)} = \frac{\lambda}{\pi} \mathrm{e}^{-2\lambda|z|^2}}$$

In the limit $\boxed{\lambda \to \infty}$ this gives a Dirac's $\boxed{\delta}$. Note that the normalization is for the integral of the functions (a probability density), not for its square, it is not a wave function.

We can calculate the Fourier transform, function of the complex variables $\boxed{\xi}$, to use with the quantizer: $\boxed{\tilde{\rho} = \frac{1}{2\pi} \mathrm{e}^{-\frac{|\xi|^2}{4\lambda}}}$

Now use the Baker-Campbell-Hausdorff to rexpress the quantization map as:

$$\boxed{\widehat{\Omega}_s(\tilde{\rho}) = \int \mathrm{d}^2\xi \tilde{\rho}(\xi) \mathrm{e}^{\xi a^\dagger} \mathrm{e}^{-\bar{\xi} a} \mathrm{e}^{|\xi|^2}}$$

Ex Perform the calculation to find $\boxed{\rho_{nm}.}$

The final result is a diagonal matrix whose elements are

$$\boxed{\rho_{nn} = \frac{2\lambda}{1+\lambda}\left(\frac{1-\lambda}{1+\lambda}\right)^n}$$

The original function is Schawarzian, and in fact the matrix is trace class, the trace being one.

Ex Prove that the trace is one.

Let us look at this matrix at the variation of $\boxed{\lambda}$

When $\boxed{\lambda < 1}$ the elements are all of positive. The first element is $\boxed{\frac{2\lambda}{1+\lambda}}$, and is smaller than one.

When $\boxed{\lambda = 1}$ alle elements vanish, except $\boxed{\rho_{00} = 1}$.

When $\boxed{\lambda > 1}$ then $\boxed{\rho_{00} > 1}$ and the remaining elements alternate positive and negative values, possibly larger than one.

Let us now go back to quantum mechanics, i.e. interpret the $\boxed{x}$'s as $\boxed{p, q}$. For dimensional reasons we should have the parameter appearing as a pure number divided by $\boxed{\hbar}$.

The original function is a classical density probability for a particles at rest in the origin, with an uncertainty in position and momentum. Classically it is possible to localise the particle as sharply as I wish, and in the limit consider a particle exactly at rest in the origin.

The Weyl quantization map associates to this classical state an operator, which we may consider to be a density matrix. When $\boxed{\frac{\lambda}{\hbar} > 1}$ there is no problem, the state is a proper density matrix. Trace is $\boxed{1}$, all elements $\boxed{< 1}$.

For $\boxed{\lambda = \hbar}$ the matrix density corresponds to a pure state, the ground state of the number operator, the $\boxed{|0\rangle}$ coherent state.

Heisenberg's revenge!

As we tried to have a particle too localised, such that the the product of the uncertainties goes below $\boxed{\frac{\hbar}{2}}$, then the procedure gives an operator, but it cannot be considered a quantum state.

Going back to NCG. The fact that there is an inherent uncertainty shows in various places, for example if one $\boxed{\star}$ multiplies two gaussians centered in different points, the result is not a real functions, but the imaginary part is small, the Gaussians are spread. If they are localised both the real and imaginary part oscillate wildly.

Another case of Heisenberg's revenge is the interpretation of the UV/IR mixing: probing a direction $\boxed{x_\mu}$ for short distances (high momentum), then there is a strong uncertainty in the conjugate one, $\boxed{\theta^{\mu\nu}x_\nu}$, hence the ultraviolet in one direction becomes infrared in the other.

The fuzzy sphere as Weyl-Wigner quantization.

I will apply the Weyl-Wigner formalism to the construction of the fuzzy sphere.

A sphere is the coadjoint orbit of $\boxed{SU(2)}$, the set of points obtained acting with an element $\boxed{\xi}$ in the dual Lie algebra with the coadjoint action $\boxed{g^{-1}\xi g}$.

Impressionistically: it is the analogue of an ordinary group orbit, but for the dual of the Lie algebra. Each orbit is a homogeneous space and naturally carries a symplectic structure. This is why coadjoint orbits are so important in representation theory and quantization: they behave like classical phase spaces naturally associated with the group. Apt for quantization.

The basic tool is a system of coherent states for the group.

To define coherent states consider the unitary irreducible representations $\boxed{\widehat{R}^{(L)}(u)}$ on $\boxed{\mathbb{C}^N}$, with $\boxed{N = 2L+1}$, a basis is given by $\boxed{|L, M\rangle}$, $\boxed{M = (-L, , \ldots, L-1, L)}$.

A group element $\boxed{u}$ is a matrix we learned to calculate in quantum mechanics:

$$\boxed{\langle L, M|\widehat{R}^{(L)}(u)\,|L, M'\rangle = D^L_{MM'}(u) \quad .}$$

Fix a fiducial state, for example the highest weight $\boxed{|\psi_0\rangle = |L, L\rangle}$. If the group manifold (which I remind you is a three sphere) is parametrised by Euler angles, then $\boxed{u}$ represents a point whose "coordinates" range through $\boxed{\alpha \in [0, 4\pi) \quad , \beta \in [0, \pi) \quad , \gamma \in [0, 2\pi)}$.

For the fiducial vector, the stability subgroup $\boxed{H_{\psi_0}}$ fixes $\boxed{\beta = 0}$

Two elements $\boxed{u, u'}$ are equivalent if $\boxed{u^\dagger u' \in H_{\psi_0}}$. Then it results: $\boxed{SU(2)/H_{\psi_0} \approx S^2}$

Ex Prove that the quotient is actually a two sphere.

Identifying $\boxed{\theta = \beta}$ and $\boxed{\varphi = \alpha \mod 2\pi}$. Chosen a representative $\boxed{\tilde{u}}$ in each equivalence class of the quotient, the set of coherent states, which depend on $\boxed{N}$, is defined as:

$$\boxed{|\theta, \varphi, N\rangle = \hat{R}^{(L)}(\tilde{u})\,|L, L\rangle\ .}$$

Projecting on the basis elements, one has:

$$\boxed{\langle L, M|\theta, \varphi, N\rangle = D^L_{ML}(\tilde{u})}$$

$$\boxed{|\theta, \varphi, N\rangle = \sum_{M=-L}^{L} \left(\frac{(2L)!}{(L+M)!(L-M)!}\right)^{1/2} (\cos\theta/2)^{L+M} (\sin\theta/2)^{L-M}\, \mathrm{e}^{-\mathrm{i}\varphi M}\,|L, M\rangle.}$$

The set is nonorthgonal and overcomplete in the proper measure.

It is now possible to define a map from matrices to functions (Berezin symbol) on the sphere

$$\boxed{A^{(N)}(\theta, \varphi) = \langle \theta, \varphi, N | \widehat{A}^{(N)} | \theta, \varphi, N \rangle \ .}$$

There are operators whose symbols are the spherical harmonics of order $\boxed{J = 0, 1, \ldots, 2L}$, $\boxed{M = -J \ldots J}$: They are the fuzzy spherical harmonics

$$\boxed{\langle \theta, \varphi, N | \widehat{Y}^{(N)}_{JM} | \theta, \varphi, N \rangle = Y_{JM}(\theta, \varphi) \ ,}$$

The matrices are eigenvalues of the Laplacian, built form the generators of $\boxed{SU(2)}$ whose matrix expression is known from introductory quantum mechanics.

$$\boxed{\left[\widehat{L}^{(N)}_a, \widehat{L}^{(N)}_b \right] = \mathrm{i} \varepsilon_{abc} \widehat{L}^{(N)}_c \ ,}$$

They are eigenvectors of the fuzzy Laplacian:

$$\boxed{\nabla^2 \widehat{A}^{(N)} = \left[\widehat{L}_s^{(N)}, \left[\widehat{L}_s^{(N)}, \widehat{A}^{(N)}\right]\right]}$$

The spectrum of the fuzzy Laplacian coincides with the one of its continuum counterpart, with a cutoff at $\boxed{2L}$.

Set a Weil-Wigner correspondence between functions and matrices at level $\boxed{N}$. Given a function on the sphere expand it in spherical harmonics.

$$\boxed{f(\theta, \varphi) = \sum_{J=0}^{\infty} \sum_{M=-J}^{J} f_{JM} Y_{JM}(\theta, \varphi) .}$$

Projecting to functions which only use harmonics up to level $\boxed{J = 2L}$ is not a subalgebra. It is known (Clebsh-Gordan) that expanding the product of two spherical harmonics of level $\boxed{J, J'}$ involves harmonics of level up to $\boxed{J + J'}$.

Multiplying and then projecting gives a nonassociative algebra, more on this later (maybe).

The fuzzy harmonics are an orthogonal basis for the space of $\boxed{N \times N}$ matrices. Like their continuous counterpart are not normalised.

A matrix can be expanded in this basis as:

$$\boxed{\widehat{F}^{(N)} = \sum_{J=0}^{2L} \sum_{M=-J}^{J} F_{JM}^{(N)} \, \widehat{Y}_{JM}^{(N)} \ ,}$$

With coefficients:

$$\boxed{F_{JM}^{(N)} = \frac{\mathrm{Tr}\left[\widehat{Y}_{JM}^{(N)\dagger} \, \widehat{F}^{(N)}\right]}{\mathrm{Tr}\, \widehat{Y}_{JM}^{(N)\dagger} \widehat{Y}_{JM}^{(N)}}}$$

A Weyl-Wigner map can be defined simply mapping $\boxed{\widehat{Y}_{JM}^{(N)} \ \Leftrightarrow \ Y_{JM}(\theta, \varphi)}$

The map depends on $\boxed{N}$, and is not invertible, except for a subspace. It is fuzzy because it cuts the higher harmonics, responsible for the short distance ripples.

It defines a $\boxed{\star}$ product on the functions on the sphere. The product is noncommutative but is associative. Field theories on fuzzy spheres are a very active field, and you heard more of this in other lectures.

I want to stress the spct that we obtains a discretization of the algebra, making it finite. We paid a price, the algebra is noncommutative, but we have a strong bonus: we kept unchanged the symmetry of the sphere. No lattice discretization can achieve this.

The construction is peculiar for the sphere and is not easy to export as it is. The construction based on coadjoint orbits will wrk for projective spaces, for example $\boxed{\mathbb{CP}^3 = SO(5)/U(2)}$.

Yet another possibility is to project at every level, this can be done in a judicious way in order to preserve the symetry of spheres in every dimension, although the construction is different for the even and odd cases. I will not discuss these cases, but there are rerences.

Another notable omission is the fuzzy torus. The algebra of functions on a torus is generated by two unitaries $\boxed{U_i = \mathrm{e}^{\mathrm{i}\varphi_i}}$. With $\boxed{\varphi}$ of period $\boxed{2\pi}$. Clearly the $\boxed{U}$'s commute.

Now consider the case $\boxed{U_1 U_2 = \mathrm{e}^{2\pi\mathrm{i}\theta} U_2 U_1}$. It is like having $\boxed{[\varphi_1, \varphi_2] = \mathrm{i}\theta}$, although the commutator among angles is a delicate issue.

When $\boxed{\theta = \frac{m}{n}}$ is rational, then it is possible to represent the $\boxed{U}$'s as finite (clock and shift) matrices for $\boxed{\omega = \mathrm{e}^{2\pi \mathrm{i}\theta}}$:

$$\boxed{U_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{n-1} \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.}$$

Again we retain the symmetries ($\boxed{U(1(\times U(1)}$) are preserved, and this is a fuzzy space, just like the sphere.

When $\boxed{\theta}$ is irrational this finite dimensional description does not exists, and we have the proper noncommutative torus, a very important (also historically) noncommutative space, that I have no time to discuss.

I have put several references.

Wick-Voros quantization and the Fuzzy Disc

Now I will do something terrible: I will change the normalizations! I know it should not be done, and I will use the lame excuse that in this way notations simplify. This may be true, but the real reason is that in this way I can use my papers...

Consider the quantum plane with $\boxed{z = x_1 + \mathrm{i}x_2}$, with noncommutativity $\boxed{[\hat{x}_1, \hat{x}_2] = \frac{\mathrm{i}}{2}\theta}$.
The commutator $\boxed{[a, a^\dagger] = \theta}$ is not changed.

To associate to $\boxed{f(z, \bar{z})}$ the operator $\boxed{\hat{\Omega}_v(f)}$ consider the Taylor expansion:

$$\boxed{f(\bar{z}, z) = f^{\mathsf{Tay}}_{mn} \bar{z}^m z^n \ ,}$$

to this function associate the operator

$$\boxed{\Omega_\theta(f) := \hat{f} = f^{\mathsf{Tay}}_{mn} \hat{a}^{\dagger m} \hat{a}^n.}$$

This is a close relative of the Weyl quantization we introduce earlier. It is calle Wick-Voros. Operators are normally ordered, as it is done in QFT to avoid infrared divergences.

It is obtained using a variation of the quantizer we saw earlier:

$$\boxed{\hat{D}_{\mathrm{W}}(z) = \int \mathrm{d}^2\xi \mathrm{e}^{\xi(a^\dagger - \bar{z})}\mathrm{e}^{-\bar{\xi}(a-z)}}$$

Ex Is it still true that real functions are mapped into hermitian operators?

The product can be expressed as:

$$\boxed{(f \star g)(z,\bar{z}) = f(z,\bar{z}) \exp\left[\theta\left(\overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}}\right)\right] g(z,\bar{z}),}$$

One can study field theories resulting form this product, there are differences even at the level of free propagators, but the S-matrices agree in simple cases.

The inverse map is constructed using coherent states

$$\boxed{\Omega_\theta^{-1}(\hat{f}) = f(\bar{z}, z) = \langle z | \hat{f} | z \rangle}$$

Ex Prove that the two maps are one the inverse of the other.

For the operator $\boxed{\hat{f}}$ we could have used the expansion in the eigenvectors of $\boxed{\hat{n}}$ we introduced earlier:

$$\boxed{\hat{f} = \sum_{m,n=0}^{\infty} f_{mn} |m\rangle \langle n|}$$

The coefficients are different. Applying the inverse map we just defined to $\boxed{\hat{f}}$ we obtain a new expansion for $\boxed{f}$:

$$\boxed{f(\bar{z}, z) = \mathrm{e}^{-\frac{|z|^2}{\theta}} \sum_{m,n=0}^{\infty} f_{mn} \frac{\bar{z}^m z^n}{\sqrt{n!m!\theta^{m+n}}}}$$

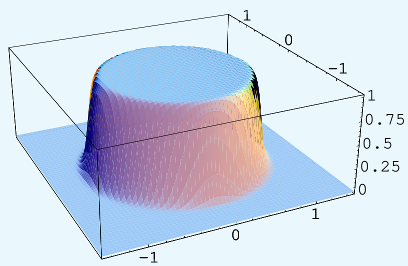
The advantage of this new expression is that now the coefficients multiply as matrices.

It makes therefore sense to truncate the algebra with a projection:

$$\boxed{P_\theta^N = \sum_{n=0}^{N} \langle z|\, n\rangle \, \langle n|\, z\rangle = \sum_{n=0}^{N} \frac{r^{2n}}{n!\theta^n} \mathrm{e}^{-\frac{r^2}{\theta}}}$$

where $\boxed{z = re^{i\phi}}$.

It approximates well the characteristic function of the disc of radius $\boxed{R = \sqrt{N\theta}}$:



The $\boxed{N = 10}$, $\boxed{R = 1}$ case.

In the limit it becomes exactly the characteristic functions.

Ex Prove that in the limit it becomes the characteritstic function.

$\boxed{P_\theta^N}$ is a projector of the algebra of functions on the plane with the $\boxed{\star}$ product: $\boxed{P_\theta^N \star P_\theta^N = P_\theta^N}$.

We have a subalgebra $\boxed{\mathcal{A}_\theta^N}$ defined as

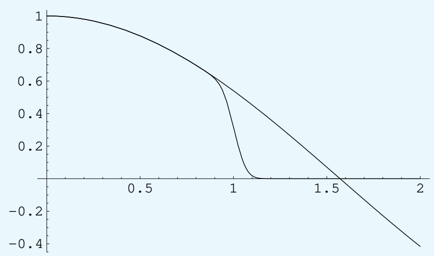
$$\boxed{\mathcal{A}_\theta^N = P_\theta^N \star \mathcal{A}_\theta \star P_\theta^N}$$

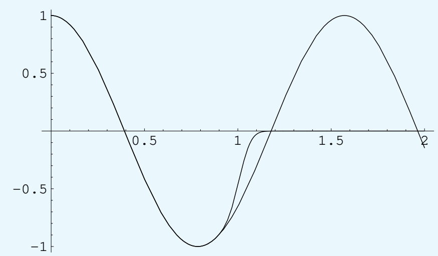
Projecting a function provides an infrared cutoff, in the sense that functions outside the disc are exponentially damped.

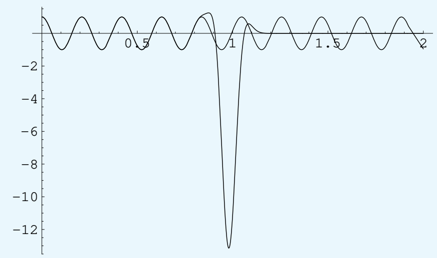
But is provides also an ultraviolet cutoff, in the sense that the map of functions become fuzzy: we have defined a fuzzy disc, But Heisenberg is ready to take again his revenge, with a form of UV/IR

If one tries to localize a function on a scale smaller than $\boxed{\sqrt{\theta}}$ large values of the function appear on the boundary

We see effect of projecting $\boxed{\cos(\alpha r)}$, we plot $\boxed{P_\theta^N \star \cos(\alpha r) \star P_\theta^N}$ for various values of $\boxed{\alpha = 1, 4, 30}$. Both functions are plotted, although inside the disc they are often indistinguishable





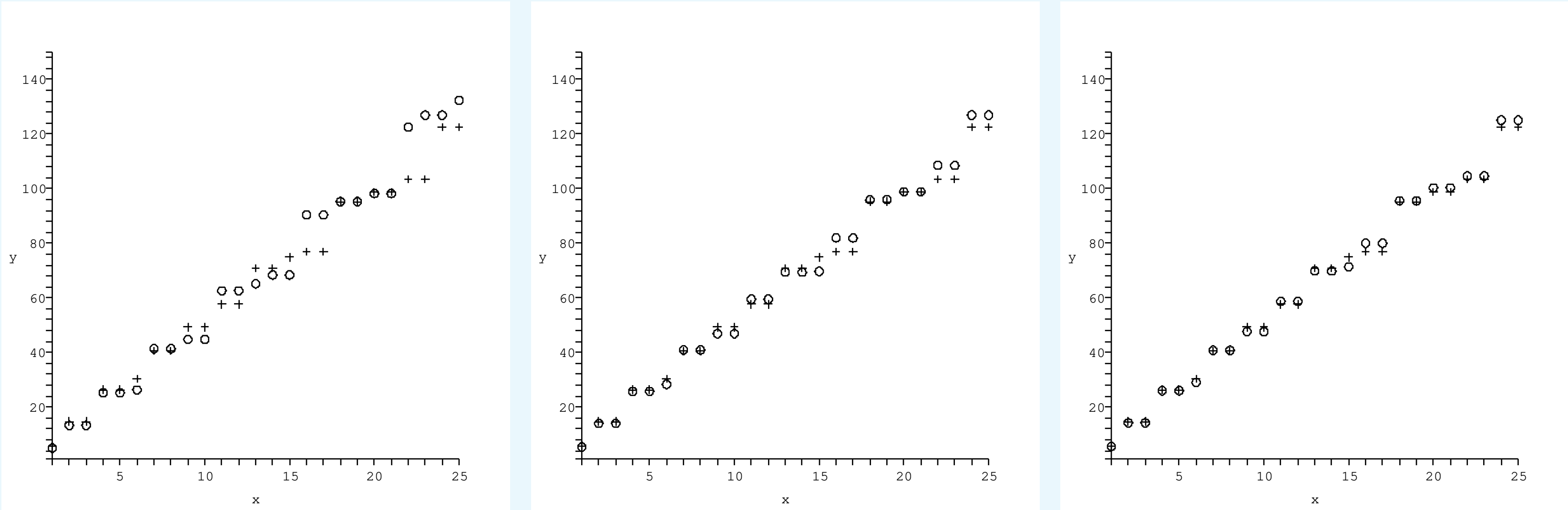


For small α the projection is just a cutoff, while for large α we see the presence of a large peak on the boundary of the disc

The problem can be circumvented defining the disc analog of the fuzzy harmonics, the fuzzy Bessel. Eigenfunctions of the fuzzy Laplacian $\boxed{\partial_z\partial_{\bar z}}$. This can achieved via the formula:

$$\boxed{\partial_z f \;=\; \frac{1}{\theta}\langle z|\left[\widehat{f},\widehat{a}^\dagger\right]|z\rangle\;, \qquad \partial_{\bar z} f \;=\; \frac{1}{\theta}\langle z|\left[\widehat{a},\widehat{f}\right]|z\rangle.}$$

this time the spectrum is approximate. Here is the comparison for $\boxed{N = 10, 20, 30}$.



Like the sphere, the fuzzy disc retains rotational symmetry, and like the sphere its construction is not easily generalizable.

It has been used to calculate aspects of field theory,a nd entanglement entropy.

So far matrices where seen as approximations, What if this sort of discretization is a real feature of the world?

This lead us to the final part of these lectures (if I still have time. . .

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