Quantum Geometry: The Spectral Approach

Lecture III: Truncated geometry

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I want to start this last lecture with some considerations about scales.

Usually the presence of an universal scale represents a big change in paradigm.

Speed of light $\boxed{c}$ acts a conversion factor, enabling to measure time as distances $\boxed{ct}$, and thus making usable spacetime in an absolute way. and it being universal makes relativity necessary.

It is also a cutoff, in the sense that the description of moving bodies is different if we may consider $\boxed{\frac{1}{c} \to 0}$.

Similar considerations can be done for $\boxed{\hbar}$. It enables to convert positions into momenta, it is a cutoff.

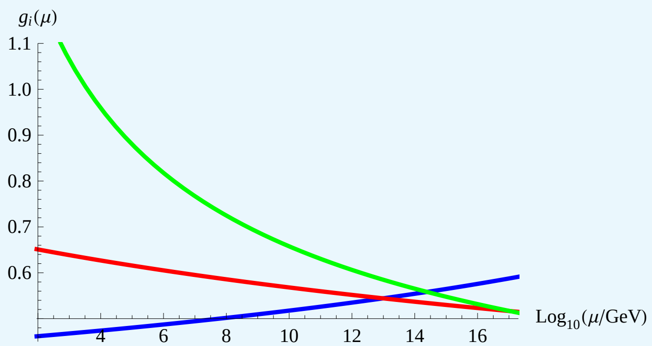
Then there the gravitational constant $\boxed{G}$. It is the strength of the gravitational interaction, in the sense that it tells how matter couples to geometry.

With the three constant we can build fundamental quantities:

- Planck length: $\boxed{\ell_{\mathsf{P}} = \sqrt{\frac{\hbar G}{c^3}} \simeq 1.616 \times 10^{-35} \text{ m.}}$
- Planck time: $\boxed{t_{\mathsf{P}} = \sqrt{\frac{\hbar G}{c^5}} = \frac{\ell_{\mathsf{P}}}{c} \simeq 5.391 \times 10^{-44} \text{ s.}}$
- Planck energy: $\boxed{E_{\mathsf{P}} = \sqrt{\frac{\hbar c^5}{G}} \simeq 1.956 \times 10^{9} \text{ J} \simeq 1.221 \times 10^{19} \text{ GeV.}}$
- Planck momentum: $\boxed{p_{\mathsf{P}} = \sqrt{\frac{\hbar c^3}{G}} = \frac{E_{\mathsf{P}}}{c} \simeq 6.525 \text{ kg m s}^{-1} \simeq 1.221 \times 10^{19} \text{ GeV}/c.}$

Do we expect something to happen for distances? Or energies? Time, or momenta?

Also from quantum field theory there are indications of a scale in which "something happens". Look at the behaviour of the coupling constants and their running.



This picture is valid in the absence of new physics, i.e. new particles and new interactions which would alter the running, in several cases (another Higgs or another $\boxed{z}$) the qualitative behaviour does not change.

The three interaction strength start from different values and almost come together at point, at a scale of about $\boxed{\Lambda = 10^{16} \text{ GeV.}}$

Then the nonabelian interactions go to asymptotic freedom, while the abelian climbs towards the Landau pole at the incredibly high energy of $\boxed{10^{53}\ \text{GeV.}}$

The current wisdom is that at $\boxed{\Lambda}$ we have to use a different quantum field theory, possibly a Grand Unified Theory, for which the standard model is just an effective theory.

One application of this principle is Chamseddine-Connes Spectral Action $\boxed{S = \operatorname{Tr}\chi\left(\frac{D_A^2}{\Lambda^2}\right)}$ I briefly discussed in the first lecture. Let me give an example of the application not to the standard model, but to the very structure of spacetime from a physical point of view. Following work I did with Kurkov and Vassilievich.

Let us analyse the role of $\boxed{\Lambda}$. Without it, the trace diverges. Field theory cannot be valid at all scales. It is itself a theory which emerges form a yet unknown quantum gravity

This points to a geometry in which the spectrum of operators like Dirac operator are truncated, i.e. the eigenvalues "saturate" at $\boxed{\Lambda}$, which appears as the top scale at which one can use QFT. One may identify this scale with $\boxed{\ell}$, but it might be different (even lower, at the unification scale).

Consider the eigenvectors $\boxed{|n\rangle}$ of $\boxed{D}$ in increasing order of the respective eigenvalue $\boxed{\lambda_n}$: $\boxed{D = \sum_0^\infty |n\rangle \lambda_n \langle n|}$.

Define $\boxed{N}$ as the maximum value for which $\boxed{\lambda_n \leq \Lambda}$. This defines the truncated Dirac operator $\boxed{\sum_0^N |n\rangle \lambda_n \langle n| + \sum_N^\infty |n\rangle \Lambda \langle n|}$.

We are effectively saturating the operator at a scale $\boxed{\Lambda}$

The role of a truncated Dirac operator, or in general wave operator, has been introduced in field theory even before the spectral action. It goes under the name of Finite Mode Regularization, Andrianov, Bonora, Fujikawa

Consider the generic fermionic action:

$$\boxed{Z = \int [d\bar{\psi}][d\psi] e^{-\langle\psi|D|\psi\rangle} \overset{\text{formally}}{=} \det D}$$

The equality is formal because the expression is divergent, and has to be regularized, for example considering $\boxed{D_\Lambda}$

One can study the renormalization flow, and note that the measure is not invariant under scale transformation, giving rise to a potential anomaly. One can add a term to the action to compensate the anomaly, and the induced term is indeed the bosonic spectral action Andrianov, Kurkov, FL

What happens if we expand the action around the high energy limit?

This can be seen explicitly Kurkov, FL, Vassilevich using an expansion in the high momentum limit introduced by Barvinsky and Vilkovisky who were able to sum all derivatives (for a decreasing exponential cutoff function).

I will not go into details, and flash the expansion.

$$\boxed{\mathrm{Tr}\,\exp\left(-\frac{D^2}{\Lambda^2}\right) \simeq \frac{\Lambda^4}{(4\pi)^2}\int d^4x\sqrt{g}\,\mathrm{tr}\,\Big[1+\Lambda^{-2}P+}$$

$$\boxed{\Lambda^{-4}\left(R_{\mu\nu}f_1\left(-\frac{\nabla^2}{\Lambda^2}\right)R^{\mu\nu}+Rf_2\left(-\frac{\nabla^2}{\Lambda^2}\right)R+\right.}$$

$$\boxed{\left.Pf_3\left(-\frac{\nabla^2}{\Lambda^2}\right)R+Pf_4\left(-\frac{\nabla^2}{\Lambda^2}\right)P+\Omega_{\mu\nu}f_5\left(-\frac{\nabla^2}{\Lambda^2}\right)\Omega^{\mu\nu}\right)\Big]+O(R^3,\Omega^3,E^3)}$$

Where $\boxed{P,\Omega,f_1,\ldots,f_5}$ are known functions of the gauge and gravitational connections $\boxed{A,\omega}$, and the Higgs field $\boxed{\phi}$.

Consider a Dirac operator containing just the relevant aspects, i.e. a bosonic fields and the fluctuations of the metric.

$$\boxed{D\!\!\!\!/ = i\gamma^\mu \nabla_\mu + \gamma_5 \phi = i\gamma^\mu(\partial_\mu + \omega_\mu + iA_\mu) + \gamma_5 \phi}$$

It is now possible to perform the B-V expansion to get the expression for the high energy spectral action:

$$\boxed{S_B \simeq \frac{\Lambda^4}{(4\pi)^2} \int d^4x \left[-\frac{3}{2} h^{\mu\nu} h_{\mu\nu} + 8\phi \frac{1}{-\partial^2} \phi + 8F^{\mu\nu} \frac{1}{(-\partial^2)^2} F_{\mu\nu} \right]}$$

compare with the usual QFT with action

$$\boxed{S[J,\varphi] = \int d^4x \left[\varphi(x) \left(\partial^2 + m^2 \right) \varphi(x) - J(x)\varphi(x) \right]}$$

To which correspond the equation of motion

$$\boxed{\left(\partial^2 + m^2 \right) \varphi(y) = J(y)}$$

Recall that in the usual, low energy case, the propagator is

$$\boxed{G(k) = \frac{1}{\left(k^2 + m^2\right)}}$$

The field at a point depends on the value of field in nearby points, and the points "talk" to each other exchanging virtual particles

Doing the analogous calculation with the B-V expansion one finds, in the opposite limit:

$$\boxed{G(x-y) \propto \begin{cases} (-\partial^2)\delta\left(x-y\right) & \text{for scalars and vectors} \\ \delta\left(x-y\right) & \text{for gravitons} \end{cases}}$$

The correlation vanishes for noncoinciding points, heuristically, nearby points "do not talk to each other".

This is a limiting behaviour, I think one has to take it as a general indication that the presence of a physical cutoff scale in momenta leads to a “non geometric phase” in which the concept of point ceases to have meaning, possibly described by a noncommutative geometry

Note that throughout this discussion I have done nothing to spacetime, I have only imposed the cutoff and used standard techniques and interpretations

Now I want to ask the question of how would the geometry look with a cutoff. I will mostrly follow a paper with Martinetti and D'Andrea, more recently picked up by work of Connes and van Suijlekom.

I will consider the ordinary geometry described in the first lecture. Let us see what happens if we truncate this geometry, using the technique as before, i.e. considering a projection operator over the lowest eigenvalues of the Dirac operator.

Recall the eigenvectors and eigenvalues of $\boxed{D}$ as $\boxed{|n\rangle}$ and $\boxed{\lambda_n}$, ordered in increasing modulus.

Define $\boxed{N}$ the largest integer such that $\boxed{|\lambda_n| < \Lambda}$, and the projector

$$\boxed{P_\Lambda = \sum_{n<N} |n\rangle \langle n|}$$

We are projecting on the "low energy" sector.

Define a projected Dirac operator

$$\boxed{D_N = P_\Lambda D P_\Lambda}$$

Since $\boxed{[P_\Lambda, D] = 0}$ one of the $\boxed{P_\Lambda}$ is redundant, I write it this way because most of what I say is valid for a generic projector.

The main difference with the untruncated case is that $\boxed{D_N}$ is a bounded operator

Likewise I can truncate the algebra and define

$$\boxed{\mathcal{A}_N = P_\Lambda \mathcal{A} P_\Lambda}$$

and analogously for the selfadjoint part $\boxed{\mathcal{S}_N}$

Note that $\boxed{\mathcal{A}_N}$ is not an algebra since $\boxed{P_\wedge f \, P_\wedge g \, P_\wedge \notin \mathcal{A}_N}$

A good example is the algebra of functions on the circle, which in the basis of the derivative (Dirac operator) is given by $\boxed{\{f_n\}, \; n \in \mathbb{Z}}$ with the convolution product

$$\boxed{(fg)_n = \sum_{n=\infty}^{\infty} f_{n-m} g_m}$$

From the operator point of view this corresponds to infinite matrices with elements on the diagonal which are equal (Töplitz matrices)

$$\boxed{P_\wedge \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & f_0 & f_1 & f_2 & \ddots \\ \ddots & f_{-1} & f_0 & f_1 & \ddots \\ \ddots & f_{-2} & f_{-1} & f_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} P_\wedge = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & f_0 & f_1 & f_2 & 0 \\ 0 & f_{-1} & f_0 & f_1 & 0 \\ 0 & f_{-2} & f_{-1} & f_0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}$$

The product of two of these matrices is not anymore a Töplitz matrix.

The space is not an algebra of operators, but it is a finite-dimensional order unit space $\boxed{\mathcal{O}_\Lambda}$, namely set of operators which do not close under multiplication, but for which it is still possible to define norm and positivity, and therefore states (in the usual way).

Connes & Van Suijlekom consider a similar notion of operator space, the important aspect is the possibility to define positivity, norm and states.

At this point we can consider the usual triple, or truncate the Dirac operator, or both $\boxed{D}$ and the algebra. And consider these three spaces and Dirac operators.

$$\boxed{(\mathcal{A}, D)\ ,(\mathcal{A}, D_\Lambda)\ ,(\mathcal{O}_\Lambda, D)}$$

They will give three notions of distance, and therefore three topologies. We are also interested in the limit $\boxed{\Lambda \to \infty}$

Let me stress the that we have not imposed a distance cutoff, but a cutoff on the energy (spectrum of $\boxed{D}$), and investigated the consequences for geometry

For lack of time I will only enumerate the results, details in the papers.

Under which conditions are these distances (strong) equivalent? Two distances equivalent if there exists positive constants $\boxed{\alpha, \beta}$ such that

$$\boxed{\alpha\, d_1(x,y) \leq d_2(x,y) \leq \beta\, d_1(x,y)\ .}$$

Strongly equivalent distances give the same topology

Truncated states, i.e. elements of $\boxed{\mathcal{O}}$ seen as elements of $\boxed{\mathcal{A}}$, are still states for the big algebra. Fancy way to say that a finite matrix can be embedded in an infinte matrix algebra

The distances between states, obtained with $\boxed{D_\Lambda}$, are strongly equivalent in the truncated space. Moreover, if $\boxed{[P_\Lambda, D] = 0}$ (true by construction) then they are strongly equivalent in the whole space.

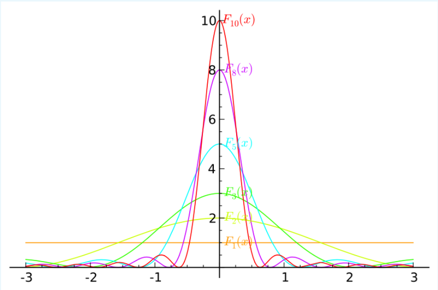
The distance on the full algebra with $\boxed{D}$ (the normal distance) is always smaller that the distance with $\boxed{D_\Lambda}$ always on the full algebra, which may be infinte. This is consistent with the earlier result with the spectral action

In this case, since the algebra is the same, the points are the same

The distance between these points, using $\boxed{D_\Lambda}$ for a generic projector can never be smaller than $1/\Lambda$, and will be infinte if the rank of $\boxed{P_\Lambda}$ is finite.

Things change if I truncate the algebra to $\boxed{\mathcal{O}}$. The topologies are still the same, but the inequalities among states are reversed.

Remember that the original pure states are not anymore usable as they are. A good example taking the Fourier expansion of a $\boxed{\delta}$, and truncate the expansion. You get the Frejer kernel



We can now calculate the distances among these truncated states, which would the "physically allowed ones".

It turn out that in this case, the distance is smaller (or equal) than the usual one for the corresponding states.

The case of the circle. Yellow: the usual distance. Magenta: The distance among approximate points with $\boxed{D}$, Blue: The distance among approximate points with $\boxed{D_\Lambda}$

It is also possible to prove that the distances converge (in the Gromov-Hausdorff sense) to the usual ones as $\boxed{\Lambda \to \infty}$.

Connes and Van Suijlekom connected truncations with tolerance.

. A tolerance relation is very similar to an equivalence relation: it is reflexive and symmetric, but it is not required to be transitive. Thus, if $\boxed{x\mathcal{R}y}$ and $\boxed{y\mathcal{R}z}$, it does not necessarily follow that $\boxed{x\mathcal{R}z}$. In other words, two points may be indistinguishable within a given tolerance, without this indistinguishability defining equivalence classes.

A relevant example comes from experiments. The error of a measuring apparatus is finite, therefore it may see as coincident two measure which differe less than the error, but the relation is not transitive, if another measure is made, further than the first, the second and third may still be superimposed, but the distance between the fIrst and third may exceed the error, and the two experimental points will be distinguishable.

Another example is directly from Mathematica

```
eps = 50*$MachineEpsilon;
a = 1;
b = a + eps;
c = b + eps;
{a == a, b == b, c == c, a == b, b == a, b == c, c == b, a == c, c == a}

(* Out:  {True, True, True, True, True, True, True, False, False}  *)
```

Due to the approximations the equality relation is actually a tolerance relation.

Start from a locally compact space $\boxed{X}$. This is a very mild requirement: finite sets of points and manifolds are locally compact.

We give $\boxed{G = X \times X}$ the obvious groupoid structure

$$\boxed{(x, y)(y, z) = (x, z)}$$

.

Recall that for a groupoid the product is not always defined. There is an inverse $\boxed{(x, y)^{-1} = (y, x)}$, and the identity (on the right or on the left) is not unique.

Completing the dual of the groupoid (linear maps) to an algebra, adding products and sums and taking the completion, get compact operators on $\boxed{L^2(X)}$.

A tolerance relation $\boxed{\mathcal{R} \subset G}$ is a symmetric subset of the groupoid containing the diagonal. Equivalently, it defines a reflexive and symmetric relation on $\boxed{X}$, but in general not a transitive one.

As an example relevant for a metric space, consider

$$\boxed{\mathcal{R}_\epsilon = \{(x, y) \in G : d(x, y) \leq \epsilon\}}$$

.

Like mathematica two points are related whenever they are closer than the resolution $\boxed{\epsilon}$. $\boxed{d(x, y) \leq \epsilon}$ and $\boxed{d(y, z) \leq \epsilon}$ do not imply $\boxed{d(x, z) \leq \epsilon}$; this is precisely why this is a tolerance relation rather than an equivalence relation.

Here the connection with truncations comes into play. The resulting system is precisely the truncated algebra. Connes and Van Suijlekom prove that:

Let $\boxed{X}$ be a finite set and $\boxed{\mathcal{R} \subseteq X \times X}$ a symmetric reflexive relation. The restriction of a pure state $\boxed{\phi \in P(\ell^2(X))}$ to $\boxed{\mathcal{R}}$ is pure if and only if the restriction of the relation $\boxed{\mathcal{R}}$ to the support $\boxed{S}$ of $\boxed{\phi}$ generates the full equivalence relation on $\boxed{S}$.

More simply, the original pure states (the points) for this non-algebra, are like points, but the identification is a tolerance.

In the case of finite partitions $\boxed{P}$ we can consider the finite-dimensional Hilbert space $\boxed{\ell^2(P)}$. In other words, we have reduced the space to a system of points.

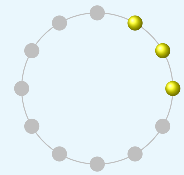
Since $\boxed{\ell^2(P) \subset L^2(X)}$, taking as vectors the characteristic functions of the intervals $\boxed{|1\rangle_u}$, and as a basis of the operator system associated with the tolerance relation the operators $\boxed{|1\rangle_u \langle 1|_v}$.

From the metric and topological point of view, this relation retains many of the properties of ordinary spaces, but the points are smeared.

A great deal has been done for finite spaces, which ultimately become graphs.

We showed what kind of nonassociative algebra arises (it is not a Jordan algebra), and connected the construction with POVMs, positive operator-valued measures.

We also studied the case of overlapping detectors and the kind of structure that emerges (a POVM, with the important condition that the total number of points and the number of detectors have coprime cardinalities).



Others things I would like to do: study the nonassociative algebra, or the operator system, or the POVM, or whatever the appropriate structure is, when I have fields that can only be constructed by superposing characteristic functions with width smaller than $\boxed{\epsilon}$.

The geometry is interesting, but can one construct a field theory on it?

A continuous field contains an infinite amount of information and depends on all points. Can one work instead with "continuous" systems containing only a finite amount of information, at the price of having tolerances rather than points?

This is still rather vague, but there are some concrete steps that could be taken, with a physicist's approach, which I may try to pursue.

The examples I made illustrative of the fact that there may be a geometry which has a built in a distance cutoff, indistinguishable from the usual one at higher scales. In a sense the geometry of low energy physics is a effective geometry.

The presence of such a cutoff, transferred in momentum space as a high energy space cutoff, would go very well with the renormalization programme.

There is powerful mathematics behind all this, as described by the work of Connes and Van Suijlekom, I will not discuss this for lack of time and competence.

The final thought with which I wish to conclude this series of lectures is that we can certainly use matrix models as useful approximations, but we can as well. entertain the thought that the world may actually be a matrix.

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