Spectral approach to quantum geometry

Exercises

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Ex1 Find, for the previous cases, examples for which $\|f^*f\| \neq \|f\|^2$ for the "wrong" norm.

Ex2 Find the states (and the pure ones) for the algebra of $n \times n$ matrices.

Ex3 Prove that the maximal ideals of this algebra correspond to the points. Hint: functions which vanish on a subset of $\mathbb{R}$ form an ideal.

Ex4 Removing the requirement that function vanish at infinity, and requiring just that they have a definite limit gives another algebra. To which topological space it corresponds?

Ex5 Prove that $\mathcal{N}_\rho$ is an ideal.

Ex6 Perform the GNS construction for $\boxed{\mathsf{Mat}(n,\mathbb{C})}$ starting from a pure state.

Ex7 Given the algebra $\boxed{\mathcal{C}_0(\mathbb{R})}$ consider the two states $\boxed{\delta_{x_0}(f) = f(x_0)}$ and $\boxed{\rho(f) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx\, e^{-x^2} a(x)}$. Find the Hilbert space in the two cases.

Ex8 Make $\boxed{\mathcal{A}^N}$ into a Hilbert module, and discuss its automorphisms.

Ex9 The algebras $\boxed{\mathbb{C}, \mathsf{Mat}(\mathbb{C}, n)}$ and $\boxed{\mathcal{K}}$, compact operators on a Hilbert space are all Morita equivalent. Find the respective bimodules.

Ex10 Take $\boxed{\mathcal{A} = \mathcal{C}(\mathbb{R})}$ and $\boxed{D = i\partial_x}$. Prove that the distance among pure states gives the usual distance among points of the line $\boxed{d(x_1, x_2) = |x_1 - x_2|.}$

Ex11 Verify that the square, the sphere and the disc have dimension $\boxed{2}$.

Ex12 The construction is made for compact Euclidean spaces. Find two good reasons for which this construction cannot be performed, without changes, for noncompact or Minkowkian space-times.

Ex13 Consider the toy model for which $\boxed{\mathcal{A}}$ is $\boxed{\mathbb{C}^2}$ represented as diagonal matrices on $\boxed{\mathcal{H} = \mathbb{C}^m \oplus \mathbb{C}^n}$, and $\boxed{D = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix}}$. Find $\boxed{D_A}$.

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Ex1 Find, for the previous cases, examples for which $\boxed{\|f^* f\| \neq \|f\|^2}$ for the "wrong" norm.

Thats an easy one! Just take $\boxed{f = \mathrm{e}^{-\frac{x^2}{2}}}$. Then

$$\boxed{\|f\|^2 = \left(\int \mathrm{d}x\, \mathrm{e}^{-x^2}\right)^2 = \pi}$$

while

$$\boxed{\|f^* f\| = \int \mathrm{d}x\, \mathrm{e}^{-2x^2} = \sqrt{\frac{\pi}{2}}}$$

The su norm in the two cases is instead 1 in both cases.

Ex2 Find the states (and the pure ones) for the algebra of $\boxed{n \times n}$ matrices.

We know the answer from quantum mechanics. Pure states correspond vectors of the Hilbert space on which the matrices act, i.e. $\boxed{n}$ dimensional vectors. Nonpure states to density matrices, i.e. hermitean matrices of trace $\boxed{1}$ and positive eigenvalues.

The first statement can be seen as follows. The algebra of matrices is also a Hilbert space with inner product $\boxed{\operatorname{Tr} b^* a}$. By Riesz theorem then any linear functional will be an element of the algebra. Then we set

$$\boxed{\phi(a) = \operatorname{Tr} \rho a}$$

$$\boxed{\phi(\mathbb{1}) = 1 \Rightarrow \operatorname{Tr} \rho = 1}$$

The fact that the matrix can be positive means that it should be Hermitean and without loss of generality we can choose a basis in which it is diagonal, and being positive with positive eigenvalues. The only way for this diagonal matrix not be be expressible as convex sum of other matrices of this kind is is to have all eigenvalues vanishing except one (which should have value one). This is a pure state, but, considering for example

$$\boxed{\rho = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}}$$

then defining $\boxed{\varphi = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}}$ we have $\boxed{\phi(a) = \text{Tr}\,\rho a = \varphi^\dagger a \varphi}$

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Ex Prove that the maximal ideals of this algebra correspond to the points. Hint: functions which vanish on a subset of $\boxed{\mathbb{R}}$ form an ideal.

Call the algebra $C_0(\mathbb{R})$. The kernel of the evaluation map at a point x is

$$I_x = \ker \delta_x = \{f \in C_0(\mathbb{R}) \mid f(x) = 0\}.$$

This is an ideal since if I multiply a function which vanishes at x with any other functions, the product will vanish at x as well. That it is maximal can be seen because it is surjective algebra homomorphism onto the complex. But intuitively, if a function vanishes in two points, it will contained in the ideal of the functions which vanish at either point, hence it is not maximal.

Since

$$C_0(\mathbb{R})/I_x \simeq \mathbb{C},$$

I_x is a maximal ideal.

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Ex Removing the requirement that function vanish at infinity, and requiring just that they have a definite limit gives another algebra. To which topological space it corresponds?

The identity is not an element of the algebra, this indicates that the resulting topological space will be compact.

We now have two more ideals, functions which vanish at plus or minus infinity. The ideals are no more the whole algebra and therefore we have two extra points. We have compactified the real line to the interval, remember that at this stage we are dealing only with topology, no metric, no other structures.

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Ex Prove that $\boxed{\mathcal{N}_\rho}$ is an ideal

This is a one-line proof: follows from

$$\boxed{\rho(a^*b^*ba) \leq \|b\|^2 \rho(a^*a)}$$

since $\boxed{\|\rho\| = \sup\{|\rho(a)| \mid \|a\| \leq 1\} = 1}$, i.e. the norm, which is one, is the supremum over the vectors of norm less than one, and $\boxed{\|ab\| \leq \|a\| \, \|b\|}$

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Ex Perform the GNS construction for $\boxed{\mathrm{Mat}(n, \mathbb{C})}$ starting from a pure state.

We will do the construction in gory detail for the two dimensional case, the generalization being straightforward.

Consider the matrix algebra $\mathrm{Mat}(n, \mathbb{C})$ with the two pure states

$$\phi_1\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11} \ , \quad \phi_2\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{22} \ .$$

The ideals of elements of 'vanishing norm' of the states ϕ_1, ϕ_2 are, respectively,

$$\mathcal{N}_1 = \left\{\begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix}\right\} \ , \qquad \mathcal{N}_2 = \left\{\begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix}\right\} \ .$$

The associated Hilbert spaces are then found to be

$$\mathcal{H}_1 = \left\{ \begin{bmatrix} x_1 & 0 \\ x_2 & 0 \end{bmatrix} \right\} \simeq \mathbb{C}^2 = \left\{ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} , \ \langle X | \, X' \rangle = x_1^* x_1' + x_2^* x_2' \ .$$

$$\mathcal{H}_2 = \left\{ \begin{bmatrix} 0 & y_1 \\ 0 & y_2 \end{bmatrix} \right\} \simeq \mathbb{C}^2 = \left\{ X = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\} , \ \langle Y | \, Y' \rangle = y_1^* y_1' + y_2^* y_2' \ .$$

As for the action of an element $A \in \mathbf{M}_2(\mathbb{C})$ on $\mathcal{H}_1$ and $\mathcal{H}_2$, we get

$$\pi_1(A) \begin{bmatrix} x_1 & 0 \\ x_2 & 0 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 & 0 \\ a_{21}x_1 + a_{22}x_2 & 0 \end{bmatrix} \equiv A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ ,$$

$$\pi_2(A) \begin{bmatrix} 0 & y_1 \\ 0 & y_2 \end{bmatrix} = \begin{bmatrix} 0 & a_{11}y_1 + a_{12}y_2 \\ 0 & a_{21}y_1 + a_{22}y_2 \end{bmatrix} \equiv A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \ .$$

The equivalence of the two representations is provided by the off-diagonal matrix

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ,$$

which interchanges 1 and 2 : $U\xi_1 = \xi_2$. Using the fact that for an irreducible representation any nonvanishing vector is cyclic, from (1) we see that the two representations can be identified.

The procedure generalize to arbitrary n, and also, with little work, to compact operators.

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Ex Given the algebra $\boxed{\mathcal{C}_0(\mathbb{R})}$ consider the two states $\boxed{\delta_{x_0}(f) = f(x_0)}$ and $\boxed{\rho(f) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx\, e^{-x^2} a(x)}$. Find the Hilbert space in the two cases.

In the first case $\mathcal{N}_\delta$ is composed of all functions which vanish at x_0. Two function belong to the same equivalence class of $\mathcal{C}_0(\mathbb{R})/\mathcal{N}_\delta$ if they differ by any function which vanishes at x_0. Therefore they must have the same value in x_0, and we can identify the class of equivalence with this value. Therefore the quotient is simply $\mathbb{C}$. Note that by the same token the vaue of the function at a point is also an irreducible representation of the algebra. Since the algebra is commutative the only IRR are indeed $\mathbb{C}$.

In the second case $\mathcal{N} = \emptyset$ since $\rho(f^* f) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx\, e^{-x^2} |f(x)|^2$ cannot be zero if $a \neq 0$. We then have a faithful (but reducible) representation.

This state is clearly not pure, for example:

$$\rho(f) = \frac{1}{2}\rho_1(f) + \frac{1}{2}\rho_2(f) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} dx\, \mathrm{e}^{-x^2} f(x) + \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} dx\, e^{-x^2} f(x)$$

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Ex Make $\boxed{\mathcal{A}^N}$ into a Hilbert module, and discuss its automorphisms

This is almost trivial. An element of $A \in \mathcal{A}_N$ is just an N-ple of element of the algebra $\{A_i\}$ and I can define a right (left) module as

$$(aA)_i = aA_i \ ; \ (Aa)_i = A_i a$$

The inner product can be easily defined

$$\langle a, b\rangle_{\mathcal{A}} = \sum_i a_i^\dagger b_i$$

The corresponding norm is

$$\|(a_1, \cdots, a_N)\|_{\mathcal{A}} := \left\| \sum_{k=1}^{n} a_k^* a_k \right\| \ .$$

That $\mathcal{A}^N$ is complete in this norm is a consequence of the completeness with respect to its norm. Parallel to the situation of the previous example, when the algebra is unital, one finds that $\mathsf{End}_{\mathcal{A}}(\mathcal{A}^N) \simeq \mathsf{End}^0_{\mathcal{A}}(\mathcal{A}^N) \simeq \mathbb{M}_n(\mathcal{A})$, acting on the left on $\mathcal{A}^N$. The isometric isomorphism $\mathsf{End}^0_{\mathcal{A}}(\mathcal{A}^N) \simeq \mathbb{M}_n(\mathcal{A})$ is now given by

$$\mathsf{End}^0_{\mathcal{A}}(\mathcal{A}) \ni |(a_1, \cdots, a_N)\rangle \langle (b_1, \cdots, b_N)| \;\; \mapsto \;\; \begin{pmatrix} a_1 b_1^* & \cdots & a_1 b_N^* \\ \vdots & & \vdots \\ a_N b_1^* & \cdots & a_N b_N^* \end{pmatrix} ,$$
$$\forall \; a_k, b_k \in \mathcal{A} \; ,$$

which is then extended by linearity.

The automorphisms of this module are simple $N \times N$ matrices with entries in the element of the algebra. If $\mathcal{A}$ is the algebra Note that for $\mathcal{A} = \mathsf{Mat}(n, \mathbb{C})$ the same space is also a Hilbert *space*, i.e. a Hilbert module over $\mathbb{C}$

$$\langle a, b\rangle_{\mathbb{C}} = \sum_i \mathsf{Tr}\, a_i^\dagger b_i$$

While for $\mathcal{A} = C_0(M)$

$$\langle a, b\rangle_{\mathbb{C}} = \sum_i \int \mathsf{d}\mu\, a_i^\dagger b_i$$

and the completion of the integral gives L^2.

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Ex The algebras $\boxed{\mathbb{C}, \mathrm{Mat}(\mathbb{C}, n)}$ and and $\boxed{\mathcal{K}}$, compact operators on a Hilbert space are all Morita equivalent. Find the respective bimodules

For any integer n the algebras $\mathbb{M}_n(\mathbb{C})$ and $\mathbb{C}$ are Morita equivalent and the equivalence $\mathbb{M}_n(\mathbb{C})$-$\mathbb{C}$ bimodule is just $\mathcal{E} = \mathbb{C}^n$. The left action of $\mathbb{M}_n(\mathbb{C})$ on $\mathcal{E}$ is the usual matrix action on a vector, while $\mathbb{C}$ acts on the right on each component of the vector. The hermitian products for any two vectors $u = (u_i)$ and $v = (v_i)$ are $\langle v|\, w\rangle_{\mathbb{C}} := \sum_{i=1}^n \bar{v}_i w_i$ and ${}_{\mathbb{M}_n(\mathbb{C})}\langle u, v\rangle := |u\rangle\langle v|$ which reads $\bar{u}_i v_j$ in components. It is immediate to verify relation (**??**):

$$
{}_{\mathbb{M}_n(\mathbb{C})}\langle v, w\rangle\, u = |v\rangle\langle w|\, u = v\,\langle w|\, u\rangle_{\mathbb{C}} \ .
$$

In components all of the above expressions are simply $\sum_j \bar{v}_i w_j u_j$. The module $\mathcal{E} = \mathbb{C}^n$ is free as a right $\mathbb{C}$-module while it is projective (of finite type) as a left $\mathbb{M}_n(\mathbb{C})$-module and $\mathbb{C}^n = \mathbb{M}_n(\mathbb{C})p$ where $p = |v\rangle\langle v|$ is any rank-one projection.

Generalizing the procedure one shows that the algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a separable Hilbert space $\mathcal{H}$ is Morita equivalent to the algebra $\mathbb{C}$ with $\mathcal{H}$ the equivalence $\mathcal{K}(\mathcal{H})$-$\mathbb{C}$ bimodule and hermitian products ${}_{\mathcal{K}(\mathcal{H})}\langle v, w\rangle := |v\rangle \langle w|$ and $\langle v|\, w\rangle_{\mathbb{C}} := \langle v|\, w\rangle_{\mathcal{H}}$. Again $\mathcal{H} = \mathcal{K}(\mathcal{H})p$ where $p = |v\rangle \langle v|$ is any rank-one projection but now $\mathcal{H}$ is not finite generated (i.e. finite dimensional) over $\mathbb{C}$.

If M is a locally compact Hausdorff topological space, then from the previous considerations follows that for any integer n the algebra $\mathbb{M}_n(\mathbb{C}) \otimes C_0(M) \simeq \mathbb{M}_n(C_0(M))$ is Morita equivalent to the algebra $C_0(M)$.

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Ex Take $\boxed{\mathcal{A} = \mathcal{C}(\mathbb{R})}$ and $\boxed{D = i\partial_x}$. Prove that the distance among pure states gives the usual distance among points of the line $\boxed{d(x_1, x_2) = |x_1 - x_2|}$

Given a function $a(x)$ of the algebra we should consider the supremum of $|a(x_1) - a(x_2)$ subject to the condition on the norm of $[D, a] \leq 1$ is

$$\|[D, a]\| = \| \sup_x \|a'(x)\|$$

clearly the supremum is attained for any function which ha $|a'(x) = 1|$ in the interval $[x_1, x_2]$, and the behaviour of the function outside is irrelevant (it just just be such that it eventually goes to zero at infinity).

In more dimensions, with genric γ's with $\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}$ the calculation is more involved for g generic, it goes through a Lipschitz norm, and in the end one obtains the usual geodesic distance given by the metric tensor.

back

Ex erify that the square, the sphere and the disc have dimension $\boxed{2}$.

Consider a first square of side L,

$$0 < x < L, \qquad 0 < y < L,$$

with Dirichlet boundary conditions. The eigenfunctions are

$$\phi_{mn}(x,y) = \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right), \qquad m,n = 1,2,\dots,$$

and the eigenvalues of $-\Delta$ are

$$\boxed{\lambda_{mn} = \frac{\pi^2}{L^2}(m^2+n^2).}$$

The condition $\lambda_{mn} \le \omega$ is

$$m^2 + n^2 \le \frac{L^2}{\pi^2}\omega.$$

Thus $N(\omega)$ is asymptotically the number of lattice points in the positive quadrant inside a circle of radius

$$R_\omega = \frac{L}{\pi}\sqrt{\omega}.$$

Hence

$$N(\omega) \sim \frac{1}{4}\pi R_\omega^2 = \frac{1}{4}\pi \frac{L^2}{\pi^2}\omega = \boxed{\frac{L^2}{4\pi}\omega}.$$

Consider the two-sphere S_R^2 of radius R. The eigenfunctions are the spherical harmonics

$$Y_{\ell m}, \qquad \ell = 0, 1, 2, \ldots, \qquad m = -\ell, \ldots, \ell,$$

and

$$-\Delta Y_{\ell m} = \lambda_\ell Y_{\ell m},$$

with

$$\boxed{\lambda_\ell = \frac{\ell(\ell+1)}{R^2}.}$$

The multiplicity of λ_ℓ is

$$g_\ell = 2\ell + 1.$$

The number of eigenvalues up to level ℓ is therefore

$$N_\ell = \sum_{j=0}^{\ell} (2j+1) = (\ell+1)^2.$$

For large ℓ,

$$\lambda_\ell \sim \frac{\ell^2}{R^2},$$

so that

$$\ell^2 \sim R^2\omega.$$

Consequently,

$$N(\omega) \sim \ell^2 \sim \boxed{R^2\omega}.$$

Consider a disc of radius R. In polar coordinates,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

With Dirichlet boundary conditions at $r = R$, the eigenfunctions are

$$\phi_{mn}(r, \theta) = J_{|m|}\left(\frac{j_{|m|,n}}{R} r\right) e^{im\theta},$$

where

$$m \in \mathbb{Z}, \qquad n = 1, 2, \ldots,$$

and $j_{\nu,n}$ denotes the n-th positive zero of the Bessel function J_ν.

The corresponding eigenvalues are

$$\boxed{\lambda_{mn} = \frac{j_{|m|,n}^2}{R^2}.}$$

Thus the counting function is

$$N(\omega) = \#\left\{(m, n) : j_{|m|,n} \leq R\sqrt{\omega}\right\},$$

including the twofold degeneracy associated with m and $-m$ for $m \neq 0$.

The asymptotic distribution of the Bessel zeros gives

$$N(\omega) \sim \frac{R^2}{4}\omega.$$

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Ex The construction is made for compact Euclidean spaces. Find two good reasons for which this construction cannot be performed, without changes, for noncompact or Minkowkian spacetimes..

On a noncompact space a derivative operator such as the original Dirac operator does not have a discrete spectrum, and therefore our construction based on the eigenvalues of D will not count. This a technical infrared problem, and on can circumvent it putting the system "in a box".

A Minkowkian noncompact space in somewhat unnatural, and not compatible with causality, what happens to the light cones? Things can be done, for example defining space with pseudonorms, called Krein spaces. back

Ex Consider the toy model for which $\boxed{\mathcal{A}}$ is $\boxed{\mathbb{C}^2}$ represented as diagonal matrices on $\boxed{\mathcal{H} = \mathbb{C}^m \oplus \mathbb{C}^n}$, and $\boxed{D = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix}}$. Find $\boxed{D_A}$

The exercise really has to do with finding one forms. We will solve it using some more sophisticated mathematics, which will help us introduce some concepts we overlooked in the lecture.

Consider a space made of two points $Y = \{1, 2\}$. The algebra $\mathcal{A}$ of continuous functions is the direct sum $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$ and any element $f \in \mathcal{A}$ is a pair of complex numbers (f_1, f_2), with $f_i = f(i)$ the value of f at the point i. An even spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is constructed as follows. The finite dimensional Hilbert space $\mathcal{H}$ is a direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and elements of $\mathcal{A}$

act as diagonal matrices

$$\mathcal{A} \ni f \mapsto \begin{pmatrix} f_1 \mathbb{1}_{\dim \mathcal{H}_1} & 0 \\ 0 & f_2 \mathbb{1}_{\dim \mathcal{H}_2} \end{pmatrix}$$

There is a natural grading operator γ given by

$$\gamma = \begin{pmatrix} \mathbb{1}_{\dim \mathcal{H}_1} & 0 \\ 0 & -\mathbb{1}_{\dim \mathcal{H}_2} \end{pmatrix} \; .$$

An operator D – being required to anticommute with γ – must be an off-diagonal matrix,

$$D = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix} \; , \quad M \in \mathrm{Lin}(\mathcal{H}_2, \mathcal{H}_1) \; .$$

With $f \in \mathcal{A}$, one finds for the commutator

$$[D, f] = (f_2 - f_1) \begin{pmatrix} 0 & M \\ -M^* & 0 \end{pmatrix} \; ,$$

and for its norm, $\|[D, f]\| = |f_2 - f_1|\lambda$ with λ the largest eigenvalue of the matrix $|M| = \sqrt{M^* M}$. Therefore, the noncommutative distance between the two points of the space is found to be

$$d(1, 2) = \sup\{|f_2 - f_1| \mid \|[D, f]\| \leq 1\} = \frac{1}{\lambda} .$$

Since D is a finite hermitian matrix, this geometry is just '0-dimensional' and the only available trace is ordinary matrix trace.

It turns out that for this space, and for more general discrete spaces as well, it is not possible to introduce a real structure which fulfills all the requirements. It seems that it is not possible to satisfy the first order condition.

We now construct the exterior algebra on this two-point space. The space $\Omega^1\mathcal{A}$ of universal 1-forms can be identified with the space of functions on $Y \times Y$ which vanish on the diagonal.

Since the complement of the diagonal in $Y \times Y$ is made of two points, namely the pairs $(1,2)$ and $(2,1)$, the space $\Omega^1\mathcal{A}$ is two-dimensional and a basis is constructed as follows. Consider the function e defined by $e(1) = 1, e(2) = 0$; clearly, $(1-e)(1) = 0$, $(1-e)(2) = 1$. A possible basis for the 1-forms is then given by*

$$e\delta e \ , \quad (1-e)\delta(1-e) \ ,$$

and their values are

$$\begin{aligned} (e\delta e)(1,2) &= -1 \ , \quad ((1-e)\delta(1-e))(1,2) = 0 \\ (e\delta e)(2,1) &= 0 \ , \quad\ \ ((1-e)\delta(1-e))(2,1) = -1 \ . \end{aligned}$$

where I defined $\delta e \equiv [D, e]$.

*Here I am oversimplifying a construction based on cyclic cohomolgy.

Any 1-form can be written as $\alpha = \lambda e \delta e + \mu(1-e)\delta(1-e)$, with $\lambda, \mu \in \mathbb{C}$. One immediately finds that

$$e[D,e] = \begin{pmatrix} 0 & -M \\ 0 & 0 \end{pmatrix} , \tag{1}$$

$$(1-e)[D,1-e] = \begin{pmatrix} 0 & 0 \\ -M^* & 0 \end{pmatrix} , \tag{2}$$

and a generic 1-form $\alpha = \lambda e \delta e + \mu(1-e)\delta(1-e)$ is

$$\alpha = -\begin{pmatrix} 0 & \lambda M \\ \mu M^* & 0 \end{pmatrix} .$$

Hermitean one forms have $\mu = \bar{\lambda}$.

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