From the Saddle Point Equation to the Eigenvalue Distribution

A Brief Guide to Resolvents and Riemann-Hilbert Problems

1. From Discrete to Continuum Saddle Point Equation

For a finite number of eigenvalues N, minimizing the energy of the associated Coulomb gas yields a set of coupled algebraic equations. Assuming standard large-N scaling of the potential $V(x)$, the discrete equilibrium condition for the highest probability configuration is:

$$V'(x_i) = \frac{2}{N} \sum_{j \neq i} \frac{1}{x_i - x_j}, \quad \text{for } i = 1, \dots, N \tag{1}$$

To transition to the macroscopic regime ($N \to \infty$), we introduce the **empirical eigenvalue density**:

$$\rho_N(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i) \tag{2}$$

As N becomes very large, the discrete energy levels pack densely into continuous intervals, and the empirical measure $\rho_N(x)$ smoothly approximates a continuous probability density function, $\rho(x)$.

We can rewrite the discrete sum as an integral over this density. Because the self-interaction term ($j = i$) is explicitly excluded from the discrete sum, the singularity at $x = y$ in the continuum must be avoided. This physical exclusion naturally translates mathematically to taking the **Cauchy principal value** ($\mathcal{P}$) of the integral. Thus, the continuum limit yields the macroscopic **saddle point equation**:

$$V'(x) = 2\mathcal{P} \int_{\Gamma} \frac{\rho(y)}{x - y}\, dy, \quad \text{for } x \in \Gamma \tag{3}$$

where $\Gamma = \mathrm{supp}(\rho)$ is the support of the eigenvalue density. Since $\rho(x)$ is a probability density, it must also satisfy $\int_{\Gamma} \rho(x) dx = 1$ and $\rho(x) \geq 0$.

2. The Resolvent (Stieltjes Transform)

To solve the integral equation (3), we introduce a powerful complex analytic tool known as the **resolvent** (or Stieltjes transform) of the empirical eigenvalue distribution:

$$W(z) = \int_{\Gamma} \frac{\rho(y)}{z - y}\, dy, \quad \text{for } z \in \mathbb{C} \setminus \Gamma \tag{4}$$

The resolvent has two fundamental properties:

1. **Analyticity:** $W(z)$ is an analytic function everywhere on the complex plane except on the support Γ, where it has branch cuts.

2. **Asymptotics:** As $|z| \to \infty$, expanding the denominator yields $W(z) \sim \frac{1}{z} \int \rho(y) dy = \frac{1}{z}$.

3. The Scalar Riemann-Hilbert Problem

To connect $W(z)$ to the saddle point equation, we evaluate the resolvent infinitesimally close to the real axis. Using the Sokhotski-Plemelj theorem, for $x \in \Gamma$:

$$W(x \pm i\epsilon) = \mathcal{P} \int_\Gamma \frac{\rho(y)}{x-y}\, dy \mp i\pi\rho(x) \tag{5}$$

Adding and subtracting these boundary values yields two critical relations:

$$W(x+i\epsilon) + W(x-i\epsilon) = 2\mathcal{P} \int_\Gamma \frac{\rho(y)}{x-y}\, dy = V'(x) \quad \text{(using Eq. 3)} \tag{6}$$

$$W(x-i\epsilon) - W(x+i\epsilon) = 2\pi i\rho(x) \tag{7}$$

Equation (6) transforms our integral equation into a **scalar Riemann-Hilbert (RH) problem**: Find a function $W(z)$ analytic on $\mathbb{C}\backslash\Gamma$ with the specified asymptotic behavior ($W(z) \sim 1/z$) and the jump condition $W_+(x) + W_-(x) = V'(x)$ across the cut(s) Γ. Once $W(z)$ is found, the eigenvalue density is simply extracted from the discontinuity across the cut via Equation (7):

$$\rho(x) = -\frac{1}{2\pi i}\Big(W(x+i\epsilon) - W(x-i\epsilon)\Big) \tag{8}$$

4. Choice of Cuts (One-cut vs. Multi-cut)

The support Γ generally consists of a finite union of disjoint intervals $[a_k, b_k]$ (the "cuts").

- **One-cut solution:** If the potential $V(x)$ is strictly convex (e.g., $V(x) = x^2$), the eigenvalues accumulate in a single interval $[a, b]$. The resolvent typically involves a single square root branch cut, $\sqrt{(z-a)(z-b)}$.
- **Multi-cut solutions:** For non-convex potentials (e.g., a double-well potential $V(x) = x^4 - tx^2$), eigenvalues can accumulate in multiple disconnected intervals. The resolvent will contain higher-order algebraic singularities like $\sqrt{\prod_{k=1}^{K}(z-a_k)(z-b_k)}$, and their exact edges are determined by enforcing the unit normalization and continuity of the measure.

Example: The Gaussian Ensemble (Wigner Semicircle)

For the standard Gaussian ensemble, the confining potential is quadratic: $V(x) = x^2$.

1. **Setup:** The derivative is $V'(x) = 2x$. Assuming a single-cut solution symmetric around the origin (due to the symmetry of $V(x)$), $\Gamma = [-a, a]$. The RH jump condition becomes:

$$W(x + i\epsilon) + W(x - i\epsilon) = 2x, \quad x \in [-a, a]$$

2. **Ansatz for the Resolvent:** We seek an analytic function with a branch cut on $[-a, a]$ that satisfies the jump. We construct $W(z)$ using a polynomial part and a square-root part to match the asymptotics:

$$W(z) = z - \sqrt{z^2 - a^2}$$

Let's verify the jump condition. For $x \in [-a, a]$, the term $\sqrt{x^2 - a^2}$ becomes purely imaginary. Specifically, $z \to x \pm i\epsilon$ gives $\mp i\sqrt{a^2 - x^2}$. Thus:

$$W(x \pm i\epsilon) = x \pm i\sqrt{a^2 - x^2}$$

Adding them yields $W_+(x) + W_-(x) = 2x = V'(x)$, satisfying the saddle point condition exactly!

3. **Determining the edge a:** We must enforce $W(z) \sim 1/z$ as $z \to \infty$. Taylor expanding our resolvent for large z:

$$W(z) = z - z\left(1 - \frac{a^2}{2z^2} + \mathcal{O}(z^{-4})\right) = \frac{a^2}{2z} + \mathcal{O}(z^{-3})$$

For this to equal $1/z$, we must have $\frac{a^2}{2} = 1$, which gives $a = \sqrt{2}$. So the support is $[-\sqrt{2}, \sqrt{2}]$.

4. **Extracting the density:** Finally, we apply the Sokhotski-Plemelj extraction formula:

$$\begin{aligned}\rho(x) &= -\frac{1}{2\pi i}\Big(W(x + i\epsilon) - W(x - i\epsilon)\Big) \\ &= -\frac{1}{2\pi i}\Big((x + i\sqrt{2 - x^2}) - (x - i\sqrt{2 - x^2})\Big) = \frac{1}{\pi}\sqrt{2 - x^2}\end{aligned}$$

This is the celebrated **Wigner Semicircle Law**.

Exercise: The Quartic Potential (Gaussian + Quartic perturbation)

Consider a random matrix model with a confining potential that includes both a Gaussian term and a quartic perturbation:

$$V(x) = x^2 + gx^4 \quad (g > 0)$$

Assume that the eigenvalues accumulate on a single symmetric interval $\Gamma = [-a, a]$.

Task: Follow the steps outlined in the Gaussian example to find the resolvent $W(z)$, determine the edge of the support a, and extract the eigenvalue density $\rho(x)$.

Final Results (to check your work):

- **Resolvent Ansatz:** To match the derivative $V'(x) = 2x+4gx^3$ and the $1/z$ asymptotic behavior, the correct polynomial structure is:

$$W(z) = \left(z + 2gz^3\right) - \left(2gz^2 + 1 + ga^2\right)\sqrt{z^2 - a^2}$$

- **Support Edge:** Expanding $W(z)$ at infinity and matching the $1/z$ term to 1 yields the algebraic equation $3ga^4 + 2a^2 - 4 = 0$, giving the physical root:

$$a^2 = \frac{\sqrt{1+12g} - 1}{3g}$$

  (Note: taking $g \to 0$ recovers $a^2 \to 2$, matching the Gaussian case.)

- **Eigenvalue Density:** Using the Sokhotski-Plemelj formula across the cut gives the generalized density profile:

$$\rho(x) = \frac{1}{\pi}\left(2gx^2 + 1 + ga^2\right)\sqrt{a^2 - x^2}, \quad \text{for } x \in [-a, a]$$