Matrix Models in Theoretical Physics
Problem Sets

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Problem 1 (Eigenvalue Statistics)**.** Consider the following four different ensembles of random square $N \times N$ matrices:

a. a symmetric matrix whose entries are uniformly random 1 and -1,

b. a symmetric matrix whose entries are uniformly random 2 and -2,

c. a symmetric matrix whose entries are distributed according to the normal distribution

$$N(\mu = 0, \sigma^2 = 1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

d. a Hermitian matrix whose diagonal entries are distributed according to $N(0,1)$ and the real and imaginary parts of the off-diagonal entries according to $N(0,1/2)$.

Gradually, for increasing values of N, for all four ensembles:

- generate a large number of matrices from the ensemble,
- find their eigenvalues,
- plot the histogram of these eigenvalues,
- observe what happens to this histogram as N increases.

Problem 2 (Three-dimensional Gauss)**.** Consider a Gaussian integral in three dimensions x_1, x_2, x_3 with the following pairing rules:

$$\begin{array}{ll} x_1 - x_1 & \text{factor } 1 \\ x_2 - x_2 & \text{factor } 1 \\ x_3 - x_3 & \text{factor } 1 \\ x_1 - x_2 & \text{factor } 0 \\ x_2 - x_3 & \text{factor } a \\ x_1 - x_3 & \text{factor } b \end{array}$$

What do the matrix A^{-1} and the matrix A in the original distribution $e^{-\vec{x}^T \cdot A \cdot \vec{x}/2}$ look like? What is the value of the integral

$$\left\langle x_1^2 x_2^2 x_3^2 \right\rangle \ ?$$

Verify this result by explicit calculation of the integral

$$\int dx_1 dx_2 dx_3 \, x_1^2 x_2^2 x_3^2 \, e^{-\frac{1}{2}\vec{x}^T \cdot A \vec{x}} \ .$$

Problem 3 (Diagrams)**.** For a model of a single Hermitian random $N \times N$ matrix with the probability distribution

$$P(M) \sim e^{-\frac{1}{2}\mathrm{Tr}\left(M^2\right)}$$

explicitly calculate the expectation value

$$\left\langle \mathrm{Tr}\left(M^6\right)\right\rangle \ .$$

Problem 4 (Semicircle probability distribution)**.** Show that for the semicircle probability distribution

$$\rho(x) = \frac{2}{\pi R^2}\sqrt{R^2 - x^2}$$

the moments $\left\langle x^{2n}\right\rangle$ are given by the n-th Catalan number

$$\left\langle x^{2n}\right\rangle = \left(\frac{R}{2}\right)^{2n} C_n = \left(\frac{R}{2}\right)^{2n} \frac{1}{n+1}\binom{2n}{n} \ .$$

Read the Wikipedia article on Catalan numbers and clarify their connection to drawing planar diagrams.

Problem 5 (Diagrams reloaded)**.** For a model of a single Hermitian random $N \times N$ matrix with the probability distribution

$$P(M) \sim e^{-N\frac{1}{2}\mathrm{Tr}\left(M^2\right)}$$

explicitly calculate the expectation value

$$\left\langle \mathrm{Tr}\left(M^4\right)\mathrm{Tr}\left(M^4\right)\right\rangle \ .$$

Specifically, explicitly show that a contraction between two traces leads to a diagram that is of lower order in N. Also explicitly show that for connected diagrams a similar statement holds as before: the factor of N by which the diagram's contribution scales depends on the topology of the surface on which it can be drawn without self-intersections.

Problem 6 (Jacobian = Vandermonde determinant)**.** We write the matrix M in the form $M = U\Lambda U^\dagger$ and calculate dM_{ij}. Consider that due to the invariance of the measure, we can calculate everything around $U = \mathbb{1}$ and prove

$$\left(\prod_i dM_{ii}\right)\left(\prod_{i<j} d\mathrm{Re}M_{ij} d\mathrm{Im}M_{ij}\right) = dU\left(\prod_i d\Lambda_{ii}\right)\left[\prod_{i<j}(\lambda_i - \lambda_j)^2\right] \ .$$

Hint. It might be useful to use $UU^\dagger = \mathbb{1}$ and consider what this relation implies for dU_{ij} and $dU^\dagger_{ij}$.

Problem 7 (Resolvent and generating function)**.** For an arbitrary probability distribution $\rho(x)$, let us introduce the function

$$\omega(z) = \int_{-\infty}^{\infty} dx\, \frac{\rho(x)}{z - x} \ .$$

We call this function the resolvent. Consider that its significance lies in the fact that the coefficients of its expansion in powers of $1/z$ are the moments of the distribution ρ, i.e., that it holds

$$\omega(z) = \sum_{k=0}^{\infty} m_k \frac{1}{z^{k+1}} \ ,$$

where m_k is somehow related to the k-th moment $\langle x^k \rangle$. If we directly define the generating function

$$\phi(t) = \sum_{k=0}^{\infty} \left\langle x^k \right\rangle t^k$$

consider that it holds

$$\omega(z) = \frac{1}{z}\phi(1/z) \ .$$

Problem 8 (Saddle point)**.** Let us have functions of the form

$$\Delta_N(x) = C_N e^{-N^2 f(x)}$$

for some function $f(x)$. Consider that this sequence converges to a δ-function at the minimum of the function $f(x)$, i.e.

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} dx\, \Delta_N(x)\phi(x) = \phi(x_0) \ ,$$

where $f'(x_0) = 0$. Find the normalization constants C_N for this to hold. What if this condition is satisfied for multiple points?

Problem 9 (Discontinuity of the square root)**.** The function $f(x) = \sqrt{x}$ as a function of a real variable is well-defined for $x \geq 0$. For a complex argument $z = re^{i\theta}$, its value is

$$f(z) = \sqrt{r}e^{i\theta/2} \ ,$$

which yields two different results $\pm i\sqrt{r}$ for z approaching the negative real axis, depending on the direction from which we approach. Based on this, consider that the function

$$\omega(z) = \frac{1}{2}V'(z) - h(z)\sqrt{(z-a)(z-b)}$$

will satisfy the equation

$$\omega(\lambda + i\varepsilon) + \omega(\lambda - i\varepsilon) = V'(\lambda)$$

for appropriately chosen $a, b, h(z)$.

Problem 10 (Quartic model)**.** Consider a quartic model of a single Hermitian matrix, i.e.

$$P(M) \sim e^{-N^2\left[\frac{1}{2}\frac{1}{N}r\mathrm{Tr}\left(M^2\right) + g\frac{1}{N}\mathrm{Tr}\left(M^4\right)\right]} \ ,$$

which is the situation with $V(x) = rx^2/2 + gx^4$. Find $a, b, h(z)$ in the expression

$$\omega(z) = \frac{1}{2}V'(z) - h(z)\sqrt{(z-a)(z-b)} \ ,$$

for which this function satisfies the conditions

$$\omega(\lambda + i\varepsilon) + \omega(\lambda - i\varepsilon) = V'(\lambda) \ , \ \omega(z) \sim_{z\to\infty} \frac{1}{z} \ .$$

From the expansion of the function $\omega(z)$ in powers of $1/z$, find the first two moments of the corresponding eigenvalue density. From the relation

$$\rho(\lambda) = -\frac{1}{2\pi i}\left[\omega(\lambda + i\varepsilon) - \omega(\lambda - i\varepsilon)\right]$$

find the eigenvalue density. Under what conditions is this density well-defined in the case of $r < 0$ and in the case of $g < 0$?

Problem 11 (2-cut solution of the quartic model)**.** Consider that the function

$$\omega(z) = \frac{1}{2}V'(z) - h(z)\sqrt{(z-a)(z-b)(z-c)(z-d)}$$

can also be a solution to the Riemann-Hilbert problem

$$\omega(\lambda + i\varepsilon) + \omega(\lambda - i\varepsilon) = V'(\lambda)$$

for the quartic potential $V(x) = rx^2/2 + gx^4$. From the condition

$$\omega(z) \sim_{z\to\infty} \frac{1}{z}$$

try to find the equations determining the constants a, b, c, d. You will see that they are not uniquely determined. Consider what this means and find the solution corresponding to a symmetric support of the probability density for the eigenvalues $\rho(\lambda)$. Finally, also find this density.

Problem 12 (Phase transition)**.** Under what conditions on r, g does the solution from the previous problem exist? You should find that it exists exactly when the solution from the earlier 'Saddle point' problem does not exist. From the following relations for the free energy, consider that during a parameter change that crosses the boundary separating these two solutions, the free energy changes such that one of its derivatives is discontinuous. Which one?

$$\begin{aligned} F_{1cut} &= \frac{-r^2\delta^2 + 40r\delta}{384} - \frac{1}{2}\log\left(\frac{\delta}{4}\right) + \frac{3}{8}\,, \\ F_{2cut} &= -\frac{r^2}{16g} + \frac{1}{4}\log(4g) + \frac{3}{8}\,. \end{aligned}$$

where $\delta = a^2$ for the 1-cut solution from the 'Quartic model' problem.

Problem 13 (Simple multi-trace model)**.** Consider a quartic matrix model, which has the square of the second moment in its probability distribution

$$S(M) = \frac{1}{2}r\frac{1}{N}\mathrm{Tr}\left(M^2\right) + g\frac{1}{N}\mathrm{Tr}\left(M^4\right) + h\left[\frac{1}{N}\mathrm{Tr}\left(M^2\right)\right]^2\,.$$

Find the corresponding single-trace matrix model, find the symmetric 1-cut and 2-cut solutions, and find the phase transition line between them. What happens to the phase transition for positive and for negative N?

Bonus. Find the relation for the free energy of these solutions and investigate the properties of the phase transition.

Problem 14 (Vandermonde determinant = true determinant)**.** Show that the following equality holds

$$\prod_{i>j}(\lambda_i - \lambda_j) = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & \lambda_N & \lambda_N^2 & \dots & \lambda_N^{N-1} \end{vmatrix}\,.$$

What exactly would the linear combinations look like that transform this matrix into a matrix with polynomials $P_n = x^n - nx^{n-1} + n^2$?

Problem 15 (Gaussian Hermitian matrices)**.** In the lecture, we found that the orthogonal polynomials for the simple potential $V(x) = \frac{1}{2}rx^2$ are Hermite polynomials. From their explicit form and the relation

$$\rho(\lambda) = K(\lambda, \lambda) = \frac{1}{N}\sum_{n=0}^{N-1} \psi_n(\lambda)\psi_n(\lambda)$$

find the eigenvalue distribution sequentially for increasing values of N and observe how this distribution approaches the semicircle one.

Problem 16 (Fuzzy sphere for $N = 2, 3$)**.** For $N = 2$, the Pauli matrices σ_i and the identity matrix σ_0 form a basis for the space of all Hermitian matrices. How must the matrices be normalized so that we can declare them to be polarization tensors? What does the mapping to spherical harmonics with $l \leq 1$ look like so that everything works as it should? Find the coefficients for the decomposition of the matrix

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \tag{1}$$

which represents the most localized state around the North Pole, into the basis of polarization tensors and plot the corresponding function. For $N = 3$, the Gell-Mann matrices λ_i and the identity matrix λ_0 form a basis for the space of all Hermitian matrices. How do the previous results change in this case?

Compare the shape of the localized packet in these two examples and prepare to be amazed.

Problem 17 (I multiply, you multiply, we multiply)**.** Rotate the decomposition of the localized packet around the North Pole (1) from the previous problem using a rotation matrix into a state Φ', which is localized around some point on the equator. Then perform the product $\Phi\Phi'$ and find the corresponding function for the resulting matrix. Compare the result with the ordinary pointwise product of the functions corresponding to Φ and Φ'.

Problem 18 (**Bonus.** Fuzzy sphere for larger N)**.** Find the explicit form of the polarization tensors for a general N in terms of Christoffel symbols. Then choose a reasonably large value and repeat the procedure from the previous two problems.

Problem 19 (Equation of motion for a simple matrix model)**.** Consider a matrix model with Euclidean signature for $d = 3$ with the action

$$S = \frac{1}{g^2}\mathrm{Tr}\left([X^a, X^b][X_a, X_b] - 2m^2 X^a X_a\right) .$$

Find the condition for the minimum of this action. Show that the fuzzy sphere is also one of its solutions, for example, and find its radius as a function of the model's parameters.

Hint.

$$\frac{\partial X_{ij}}{\partial X_{kl}} = \delta_{ik}\delta_{jl} .$$

Problem 20 (Paper)**.** Read the paper

Gravity as a Quantum Effect on Quantum Space-Time

and keep track of familiar or unfamiliar words, what makes sense and what doesn't, and how well you are able to follow what is happening in the paper.